PHY481: Electromagnetism

Capacitance

Point charge near a conductor

solved by

Method of Images
Conductors in static equilibrium

Why these statements are true?
- Electric field $E$ inside a conductor is zero.
- Potential $V$ inside a conductor is a constant.
- Inside a conductor the charge density $\rho$ is zero.
- Charge $Q$ on a conductor resides only on the surface.
- A net charge $Q_{\text{net}}$ here is always paired with charge $-Q_{\text{net}}$ elsewhere.
- On a conductor either $V$ or $Q_{\text{net}}$, but not both, can be specified.
- At a conductor's surface, field component $E_t = 0$, component $E_n = \sigma/\epsilon_0$, but in force calculations, $E_n = \sigma/2\epsilon_0$ due to only “distant” charges.
- In an empty cavity in a conductor, $E = 0$ and surface $\sigma_{\text{cavity}} = 0$.
- The potential $V$ of a conductor can be set by a battery $V_B$.
- The earth is an “infinite” source of charge at a constant $V$.
- $V$ or $Q_{\text{net}}$ of conductors (and $Q_{\text{external}}$) determine $V$ everywhere.
Boundary value problems: 1-parallel plates

Big parallel plates (find potential, field & surface charge)

- The ground is a constant potential, so let \( V_1 = 0 \).
- Battery draws electrons from plate 2 and puts them on plate 1. Process “continues” until \( V_2 = V_B \).

Potential between the plates

- Between the plates the potential depends only on \( x \).

Laplace’s equation:

\[
\nabla^2 V(x) = 0
\]

Boundary conditions:

\[
V(0) = 0; \quad V(d) = V_B
\]

General solution (ODE)

\[
\nabla^2 V = \frac{d}{dx} \left( \frac{dV}{dx} \right) = 0; \quad \frac{dV}{dx} = c_1
\]

\[
V(x) = c_1 x + c_2
\]

Apply boundary conditions

\[
V(0) = c_2 = 0
\]

\[
V(d) = c_1 d = V_B; \quad c_1 = \frac{V_B}{d}
\]

Potential

\[
V(x) = V_B \frac{x}{d}
\]
Parallel plates (cont’d)

**Electric field**

\[
E = -\nabla V = -\nabla \left( V_B \frac{x}{d} \right)
\]

\[
E = -\frac{V_B}{d} \hat{i}
\]

\[
E_n = E \cdot \hat{n}
\]

**Surface 1r charge density**

\[
\sigma_{1r} = -\frac{\varepsilon_0 V_B}{d}
\]

**Surface 2ℓ charge density**

\[
\sigma_{2\ell} = +\frac{\varepsilon_0 V_B}{d}
\]

**Force on plates**

\[
F_2 = Q_2 \left( E_{2n} / 2 \right)
\]

\[
F_2 = \frac{1}{2} \frac{\sigma^2}{\varepsilon_0} A
\]
Parallel plates (cont’d)

Capacitance

Two ways to know the electric field

\[ E = \frac{\sigma}{\varepsilon_0}; \quad E = \frac{V_B}{d} \]

Capacitance is a geometrical factor relating the charge on a conductor and its potential.

\[ Q = \sigma A = \frac{\varepsilon_0 A}{d} V_B = CV_B \]

\[ C = \frac{\varepsilon_0 A}{d} \]

Energy storage

\[ U = \frac{\varepsilon_0}{2} \int E^2 d^3 x' = \frac{\varepsilon_0}{2} \frac{V_B^2}{d^2} A d = \frac{1}{2} \frac{\varepsilon_0 A}{d} V_B^2 \]

\[ U = \frac{1}{2} CV_B^2 \]
Energy change in separating the plates

Plates charged

\[ Q = CV_B = \varepsilon_0 A V_B / d_0 \]

\[ V = 0 \quad x \]

\[ V = V_B \quad -Q \]

\[ -Q \quad +Q \]

\[ 0 \quad d_0 \]

\[ V = 0 \quad x \]

\[ V = V_1 \]

\[ 0 \quad d_1 \]

\[ f \]

Disconnect battery and separate plates

Charge constant at \( Q \)

\[ Q = C_1 V_1 = \varepsilon_0 A V_1 / d_1 \]

\[ V_1 = V_B d_1 / d_0 \]

\[ E = \frac{1}{2} CV_B^2 = \frac{1}{2} \frac{\varepsilon_0 A}{d_0} V_B^2 \]

\[ E_1 = \frac{1}{2} C_1 V_1^2 = \frac{d_1}{d_0} E \]

\[ E_1' = \frac{1}{2} C_1 V_B^2 = \frac{d_0}{d_1} E \]

\[ \varepsilon \]

\[ \varepsilon_1 \]

Leave battery connected and separate plates

Voltage constant at \( V_B \)

\[ Q_1 = V_1 B = \varepsilon_0 A V_B / d_1 \]

\[ Q_1 = Qd_0 / d_1 \]
Cylindrical capacitor

Capacitance of nested long cylinders
Find charge $Q$ for the given $V = V_B$

Gauss's law between the cylinders
\[ \oint_S \mathbf{E} \cdot d\mathbf{A} = 2\pi rE(r) = \frac{\lambda_a L}{\varepsilon_0} \]

Field between the cylinders
\[ E(r) = \frac{\lambda_a}{2\pi \varepsilon_0 r} \]

Potential between the cylinders
\[ V(r) = -\int_b^r E dr = -\frac{\lambda_a}{2\pi \varepsilon_0} \int_b^r \frac{dr}{r} = \frac{\lambda_a}{2\pi \varepsilon_0} \ln\left(\frac{b}{r}\right) \]

Charge and potential relationship
\[ V_B = \frac{\lambda_a}{2\pi \varepsilon_0} \ln\left(\frac{b}{a}\right); \quad \lambda_a = \frac{2\pi \varepsilon_0}{\ln\left(\frac{b}{a}\right)} V_B \]

Capacitance per unit length
\[ Q = \lambda_a L = \frac{2\pi \varepsilon_0 L}{\ln\left(\frac{b}{a}\right)} V_B = CV_B \]

\[ C/L = \frac{2\pi \varepsilon_0}{\ln\left(\frac{b}{a}\right)} \]
Dipole potential & field on the midplane

- Finite dipole potential \( (x \neq \pm z_0 \hat{k}) \)
  \[
  V(x) = \frac{q}{4\pi \varepsilon_0} \left\{ \frac{1}{\left[ x^2 + y^2 + (z-z_0)^2 \right]^{1/2}} - \frac{1}{\left[ x^2 + y^2 + (z+z_0)^2 \right]^{1/2}} \right\}
  \]

- Dipole potential on the midplane
  \( z = 0 \), then \( V(x, y, 0) = 0 \), and \( \nabla V = 0 \), and \( \nabla^2 V = 0 \)

  Midplane is an equipotential (0)

- Is Laplace’s equation, \( \nabla^2 V(x) = 0 \) satisfied everywhere?

  Get dipole field \( \mathbf{E} \), then
  \[
  -\nabla \cdot \mathbf{E} = \nabla \cdot (\nabla V) = \nabla^2 V(x)
  \]

  Not obvious that \( \nabla \cdot \mathbf{E} = 0 \)

  \[
  \mathbf{E}(x, y, z) = \frac{q}{4\pi \varepsilon_0} \left\{ \frac{x \hat{i} + y \hat{j} + (z-z_0) \hat{k}}{\left[ x^2 + y^2 + (z-z_0)^2 \right]^{3/2}} - \frac{x \hat{i} + y \hat{j} + (z+z_0) \hat{k}}{\left[ x^2 + y^2 + (z+z_0)^2 \right]^{3/2}} \right\}
  \]

  If it is, then Laplace’s Eq. is satisfied everywhere \( (x \neq \pm z_0 \hat{k}) \)
For a dipole field
\[ \nabla^2 V(x) = -\nabla \cdot E = 0 \]

- \( C(x,y,z) \) is the position dependence of E-field of the + charge

\[
C(x,y,z) = \frac{x \hat{i} + y \hat{j} + (z-z_0) \hat{k}}{\left[ x^2 + y^2 + (z-z_0)^2 \right]^{3/2}} \quad C_x = \frac{x}{\left[ x^2 + y^2 + (z-z_0)^2 \right]^{3/2}}, \text{ etc.}
\]

\[
\frac{\partial C_x}{\partial x} = \left( \frac{1}{\left[ x^2 + y^2 + (z-z_0)^2 \right]^{3/2}} - \frac{3x^2}{\left[ x^2 + y^2 + (z-z_0)^2 \right]^{5/2}} \right)
\]

\[
\nabla \cdot C(x,y,z) = \frac{\partial C_i}{\partial x_i} = \left( \frac{3}{\left[ x^2 + y^2 + (z-z_0)^2 \right]^{3/2}} - \frac{3\left[ x^2 + y^2 + (z-z_0)^2 \right]}{\left[ x^2 + y^2 + (z-z_0)^2 \right]^{5/2}} \right)
\]

Now you can see that this really is equal to ZERO

This duplicates the discussion about the Dirac delta function

- At the charges \( x \mp z_0 \hat{k} = 0 \), and \( \nabla^2 V(x) = -\rho(x)/\varepsilon_0 \neq 0 \)
Compare dipole & charge above conducting plane

- Dipole field

\[
E(x, y, z) = \frac{q}{4\pi\varepsilon_0} \left\{ \frac{x\hat{i} + y\hat{j} + (z - z_0)\hat{k}}{\left[ x^2 + y^2 + (z - z_0)^2 \right]^{3/2}} - \frac{x\hat{i} + y\hat{j} + (z + z_0)\hat{k}}{\left[ x^2 + y^2 + (z + z_0)^2 \right]^{3/2}} \right\}
\]

- Dipole field on the midplane is normal to plane

\[
E(x, y, 0) = \frac{-qz_0\hat{k}}{2\pi\varepsilon_0\left[ r^2 + z_0^2 \right]^{3/2}}
\]

We should have known, \( E \) normal to equipotentials

1) Dipole potential satisfies Laplace’s equation. \((x \neq \pm z_0\hat{k})\)

2) Dipole potential is zero on the mid-plane.

Compare with a point charge above a grounded conducting plane.

V satisfies Laplace’s equation
V = 0, on the surface

Suggests a “method of images” for solving Laplace’s Eq.
Uniqueness of solutions to Laplace’s Eq.

Solving this problem is equivalent to solving this problem.

This problem is already solved!

E field on conductor’s surface = E field of dipole on the midplane

Conductor’s surface charge density

\[ \sigma(x, y) = \varepsilon_0 E_n(x, y) = \frac{-qz_0}{2\pi \left[ r^2 + z_0^2 \right]^{3/2}} \]

Conductor’s total charge

\[ \oint_S \sigma(x, y) d^2x' = \frac{-qz_0}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{rdr}{\left( r^2 + z_0^2 \right)^{3/2}} = -q \]