PHY481: Electromagnetism

Capacitance

Point charge near a conductor

solved by

Method of Images

Conductors in static equilibrium

Why these statements are true ?

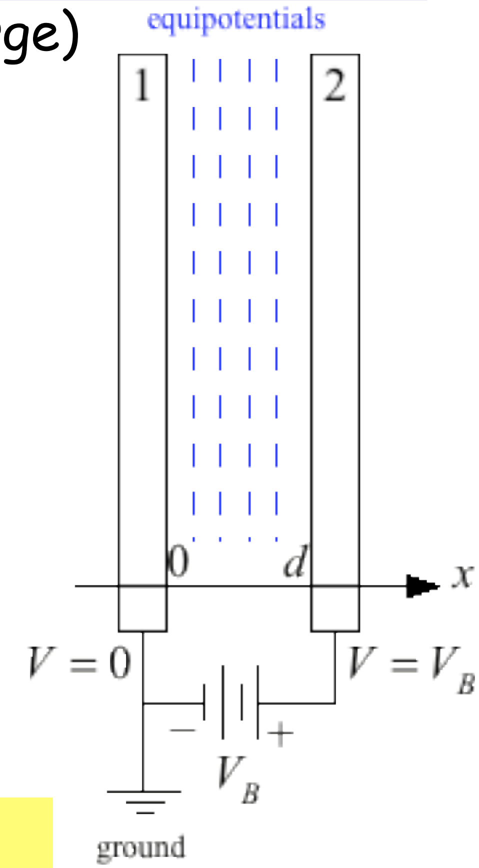
- Electric field **E** inside a conductor is zero.
- Potential V inside a conductor is a constant.
- Inside a conductor the charge density ρ is zero.
- Charge Q on a conductor resides only on the surface.
- A net charge Q_{net} here is always paired with charge $-Q_{net}$ elsewhere.
- On a conductor either V or Q_{net}, but not both, can be specified.
- At a conductor's surface, field component $E_t = 0$, component $E_n = \sigma/\varepsilon_0$, but in force calculations, $E_n = \sigma/2\varepsilon_0$ due to only "distant" charges.
- In an empty cavity in a conductor, **E** = 0 and surface $\sigma_{cavity} = 0$.
- The potential V of a conductor can be set by a battery V_B.
- The earth is an "infinite" source of charge at a constant V .
- V or Q_{net} of conductors (and $Q_{external}$) determine V everywhere.

Boundary value problems: 1-parallel plates

Big parallel plates (find potential, field & surface charge)

The ground is a constant potential, so let $V_1 = 0$.

Battery draws electrons from plate 2 and puts them on plate 1. Process "continues" until $V_2 = V_B$.



Potential between the plates

Between the plates the potential depends only on *x*.

Laplace's equation: $\nabla^2 V(x) = 0$

Boundary conditions: $V(0) = 0; \quad V(d) = V_B$

General solution (ODE)

$$\nabla^2 V = \frac{d}{dx}\left(\frac{dV}{dx}\right) = 0; \ \frac{dV}{dx} = c_1$$

$$V(x) = \underline{c_1 x + c_2}$$

Apply boundary conditions

$$V(0) = \underline{c_2 = 0}$$

$$V(d) = c_1 d = V_B; \ \underline{c_1 = \frac{V_B}{d}}$$

Potential

$$V(x) = V_B \underline{\frac{x}{d}}$$

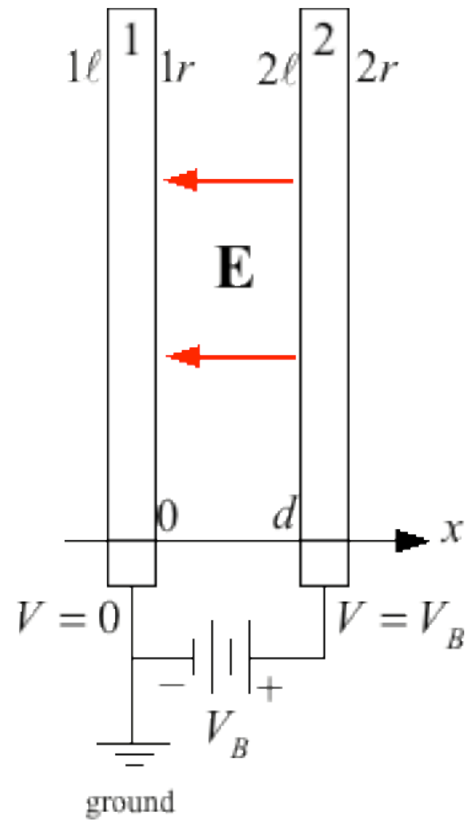
Parallel plates (cont'd)

Electric field

$$\mathbf{E} = -\nabla V = -\nabla\left(V_B \frac{x}{d}\right)$$

$$\mathbf{E} = -\frac{V_B}{d}\hat{\mathbf{i}}$$

$E_n = \mathbf{E}\cdot\hat{\mathbf{n}}$ Normal component



Surface $1r$ charge density

$$E_{1n} = \frac{\sigma_{1r}}{\varepsilon_0} = -\frac{V_B}{d}$$

$$\sigma_{1r} = -\frac{\varepsilon_0 V_B}{d}$$

Surface 2ℓ charge density

$$E_{2n} = \frac{\sigma_{2\ell}}{\varepsilon_0} = \frac{V_B}{d}$$

$$\sigma_{2\ell} = +\frac{\varepsilon_0 V_B}{d}$$

Surface charge

$$Q_{2\ell} = -Q_{1r} = \sigma A$$

$$Q_{2\ell} = \frac{\varepsilon_0 V_B A}{d}$$

Force due to Q1

$$F_2 = Q_2\left(E_{2n}/2\right)$$

Force on plates

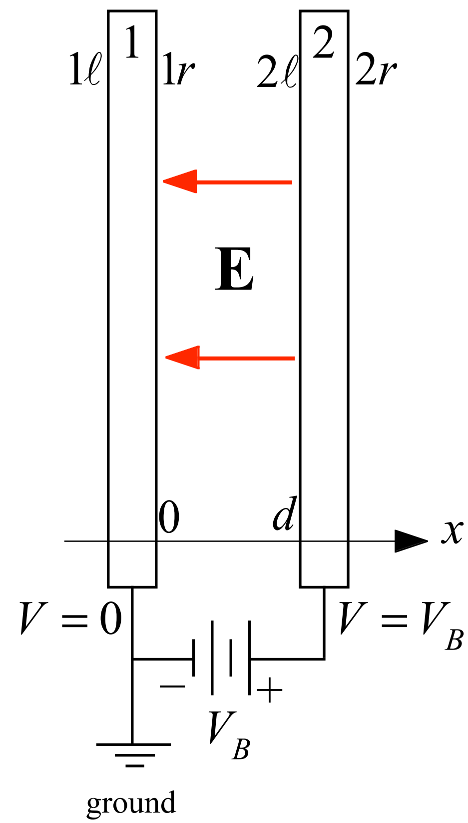
$$F_2 = \frac{1}{2}\frac{\sigma_2^2}{\varepsilon_0}A$$

Parallel plates (cont'd)

Capacitance

Two ways to know the electric field

$$E = \frac{\sigma}{\varepsilon_0}; \quad E = \frac{V_B}{d}$$



Capacitance is a geometrical factor relating the charge on a conductor and its potential.

$$Q = \sigma A = \frac{\varepsilon_0 A}{d} V_B = C V_B$$

$$C = \frac{\varepsilon_0 A}{d}$$

Energy storage

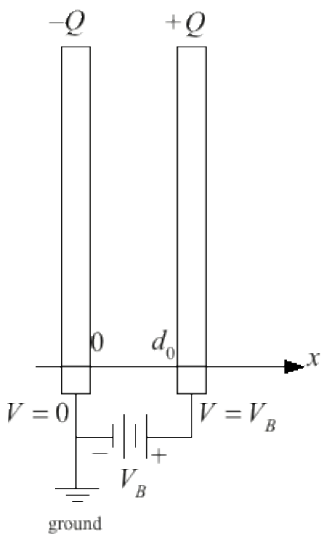
$$U = \frac{\varepsilon_0}{2} \int E^2 d^3 x'$$

$$= \frac{\varepsilon_0}{2} \frac{V_B^2}{d^2} A d = \frac{1}{2} \frac{\varepsilon_0 A}{d} V_B^2$$

$$U = \frac{1}{2} C V_B^2$$

Energy change in separating the plates

Plates charged

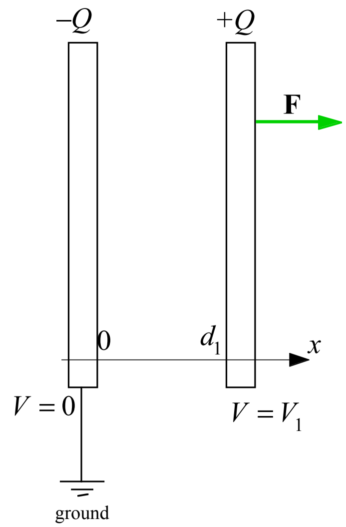


$$Q = CV_B = \varepsilon_0 A V_B / d_0$$

Stored energy

$$\mathcal{E} = \frac{1}{2} C V_B^2 = \frac{1}{2} \frac{\varepsilon_0 A}{d_0} V_B^2$$

Disconnect battery and separate plates



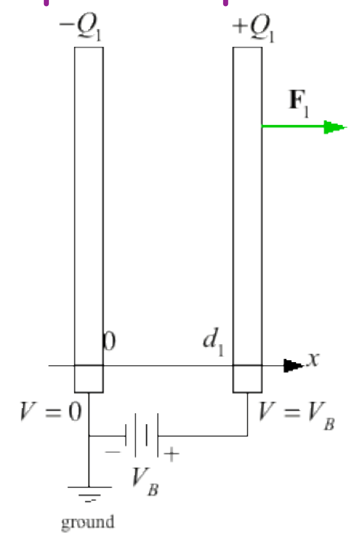
Charge constant at Q
Voltage changes to V_1

$$Q = C_1 V_1 = \varepsilon_0 A V_1 / d_1$$

$$V_1 = V_B d_1 / d_0$$

$$\mathcal{E}_1 = \frac{1}{2} C_1 V_1^2 = \frac{d_1}{d_0} \mathcal{E} \uparrow$$

Leave battery connected and separate plates



Voltage constant at V_B
Charge changes to Q_1

$$Q_1 = C_1 V_B = \varepsilon_0 A V_B / d_1$$

$$Q_1 = Q d_0 / d_1$$

$$\mathcal{E}_1' = \frac{1}{2} C_1 V_B^2 = \frac{d_0}{d_1} \mathcal{E} \downarrow ?$$

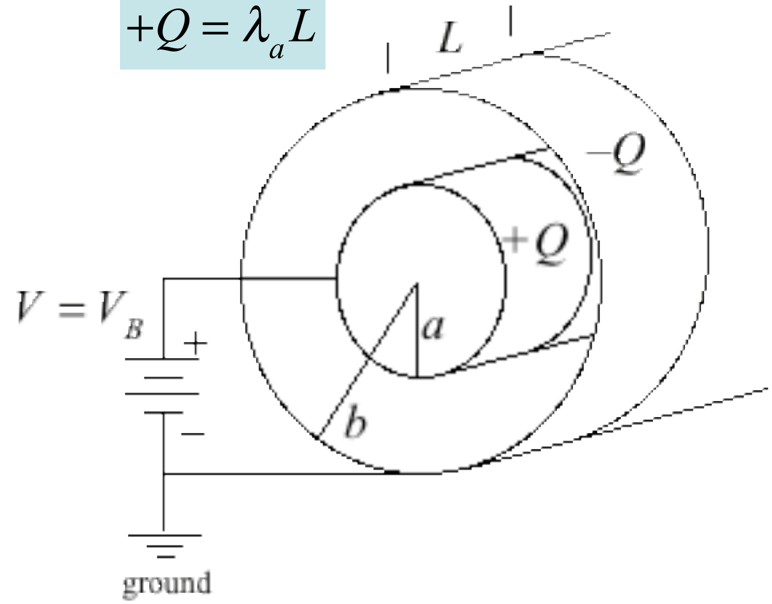
Cylindrical capacitor

Capacitance of nested long cylinders

Find charge Q for the given $V = V_B$

$$\lambda_a = \sigma_a (2\pi a)$$

$$+Q = \lambda_a L$$



Gauss's law between the cylinders

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = 2\pi r L E(r) = \frac{\lambda_a L}{\varepsilon_0}$$

Field between the cylinders

$$E(r) = \frac{\lambda_a}{2\pi\varepsilon_0 r}$$

Potential between the cylinders

$$V(r) = -\int_b^r E dr = -\frac{\lambda_a}{2\pi\varepsilon_0}\int_b^r \frac{dr}{r} = \frac{\lambda_a}{2\pi\varepsilon_0}\ln(b/r)$$

Charge and potential relationship

$$V_B = \frac{\lambda_a}{2\pi\varepsilon_0}\ln(b/a); \quad \lambda_a = \frac{2\pi\varepsilon_0}{\ln(b/a)} V_B$$

$$Q = \lambda_a L = \frac{2\pi\varepsilon_0 L}{\ln(b/a)} V_B = C V_B$$

Capacitance per unit length

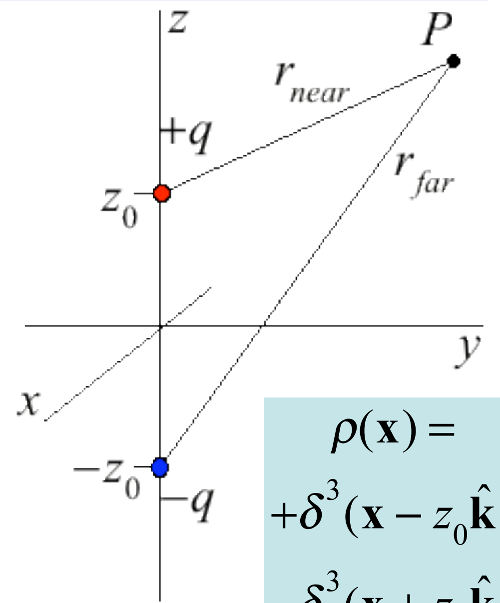
$$C/L = \frac{2\pi\varepsilon_0}{\ln(b/a)}$$

Dipole potential & field on the midplane

- Finite dipole potential $\left(\mathbf{x} \neq \pm z_0 \hat{\mathbf{k}}\right)$

$$V(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{\left[x^2+y^2+\left(z-z_0\right)^2\right]^{1/2}} - \frac{1}{\left[x^2+y^2+\left(z+z_0\right)^2\right]^{1/2}}\right\}$$

note signs



$$\rho(\mathbf{x}) = +\delta^3(\mathbf{x} - z_0\hat{\mathbf{k}}) - \delta^3(\mathbf{x} + z_0\hat{\mathbf{k}})$$

- Dipole potential on the midplane

$$z = 0, \text{ then } V(x,y,0) = 0, \text{ and } \nabla V = 0, \text{ and } \nabla^2 V = 0$$

Midplane is an equipotential (0)

- Is Laplace's equation, $\nabla^2 V(\mathbf{x}) = 0$ satisfied everywhere?

Get dipole field **E** , then $-\nabla \cdot \mathbf{E} = \nabla \cdot (\nabla V) = \nabla^2 V(\mathbf{x})$

Not obvious that $\nabla \cdot \mathbf{E} = 0$

$$\mathbf{E}(x,y,z) = \frac{q}{4\pi\varepsilon_0}\left\{\frac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+\left(z-z_0\right)\hat{\mathbf{k}}}{\left[x^2+y^2+\left(z-z_0\right)^2\right]^{3/2}} - \frac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+\left(z+z_0\right)\hat{\mathbf{k}}}{\left[x^2+y^2+\left(z+z_0\right)^2\right]^{3/2}}\right\}$$

If it is, then Laplace's Eq. is satisfied everywhere $\left(\mathbf{x} \neq \pm z_0 \hat{\mathbf{k}}\right)$

For a dipole field

$$\nabla^2 V(\mathbf{x}) = -\nabla \cdot \mathbf{E} = 0$$

- **C**(x,y,z) is the position dependence of **E**-field of the + charge

$$\mathbf{C}(x,y,z) = \frac{x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + (z - z_0)\hat{\mathbf{k}}}{\left[x^2 + y^2 + (z - z_0)^2\right]^{3/2}} \qquad C_x = \frac{x}{\left[x^2 + y^2 + (z - z_0)^2\right]^{3/2}}, \text{ etc.}$$

$$\frac{\partial C_x}{\partial x} = \left(\frac{1}{\left[x^2 + y^2 + (z - z_0)^2\right]^{3/2}} - \frac{3x^2}{\left[x^2 + y^2 + (z - z_0)^2\right]^{5/2}} \right)$$

$$\nabla \cdot \mathbf{C}(x,y,z) = \frac{\partial C_i}{\partial x_i} = \left(\frac{3}{\left[x^2 + y^2 + (z - z_0)^2\right]^{3/2}} - \frac{3\left[x^2 + y^2 + (z - z_0)^2\right]}{\left[x^2 + y^2 + (z - z_0)^2\right]^{5/2}} \right)$$

Now you can see that this really is equal to ZERO

This duplicates the discussion about the Dirac delta function

- At the charges $\mathbf{x} \mp z_0\hat{\mathbf{k}} = 0$, and $\nabla^2 V(\mathbf{x}) = -\rho(\mathbf{x})/\varepsilon_0 \neq 0$

Compare dipole & charge above conducting plane

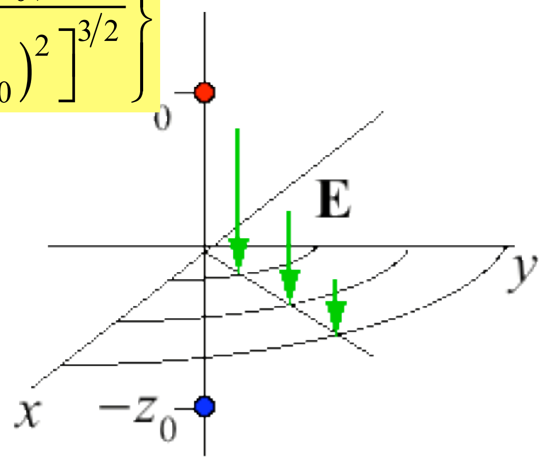
- Dipole field

$$\mathbf{E}(x,y,z)=\frac{q}{4\pi\varepsilon_0}\left\{\frac{x\,\hat{\mathbf{i}}+y\,\hat{\mathbf{j}}+(z-z_0)\hat{\mathbf{k}}}{\left[x^2+y^2+(z-z_0)^2\right]^{3/2}}-\frac{x\,\hat{\mathbf{i}}+y\,\hat{\mathbf{j}}+(z+z_0)\hat{\mathbf{k}}}{\left[x^2+y^2+(z+z_0)^2\right]^{3/2}}\right\}$$

- Dipole field on the midplane is normal to plane

$$\mathbf{E}(x,y,0)=\frac{-qz_0\,\hat{\mathbf{k}}}{2\pi\varepsilon_0\left[r^2+z_0^2\right]^{3/2}}$$

$$r^2=x^2+y^2$$

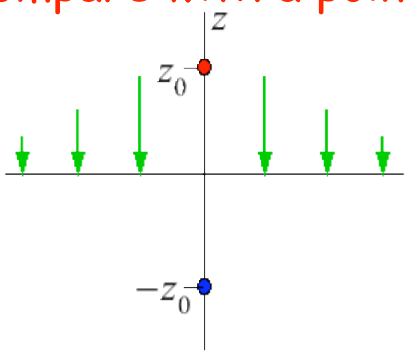


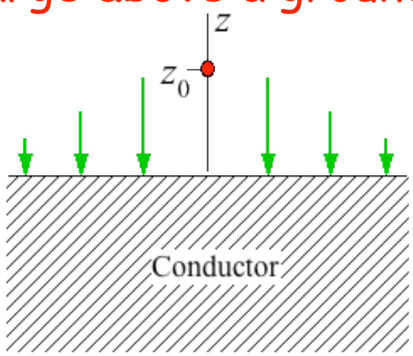
We should have known, E normal to equipotentials

1) Dipole potential satisfies Laplace's equation. $\left(\mathbf{x}\neq\pm z_0\hat{\mathbf{k}}\right)$

2) Dipole potential is zero on the mid-plane.

Compare with a point charge **above** a grounded conducting plane.





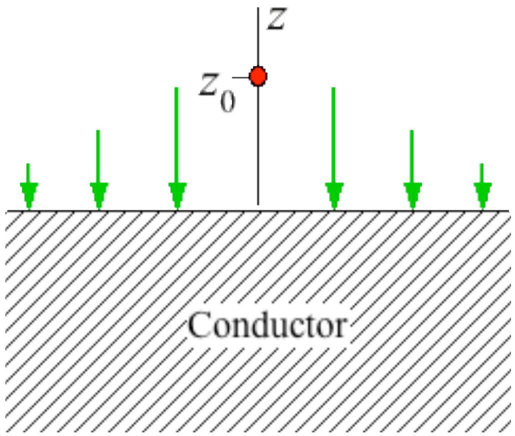
V satisfies Laplace's equation

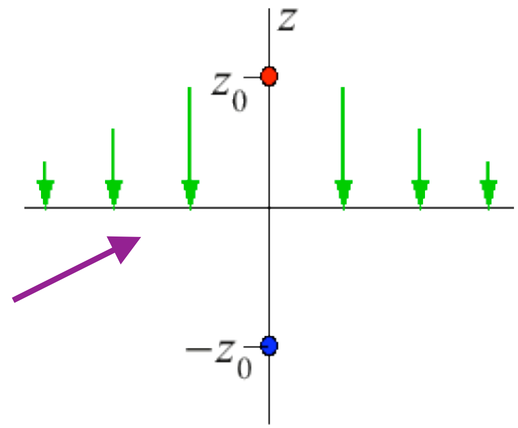
V = 0, on the surface

Suggests a "method of images" for solving Laplace's Eq.

Uniqueness of solutions to Laplace's Eq.

Solving this problem is equivalent to solving this problem





This problem is already solved!

E field on conductor's surface = **E** field of dipole on the midplane

$$\mathbf{E}(x,y,0) = \frac{-qz_0\,\hat{\mathbf{k}}}{2\pi\varepsilon_0\left[r^2+z_0^2\right]^{3/2}} \qquad r^2 = x^2 + y^2$$

Conductor's surface charge density

$$\sigma(x,y) = \varepsilon_0 E_n(x,y) = \frac{-qz_0}{2\pi\left[r^2+z_0^2\right]^{3/2}}$$

Conductor's total charge

$$\oint_S \sigma(x,y)\,d^2x' = \frac{-qz_0}{2\pi}\int_0^{2\pi} d\phi \int_0^{\infty} \frac{r\,dr}{\left(r^2+z_0^2\right)^{3/2}} = -q$$