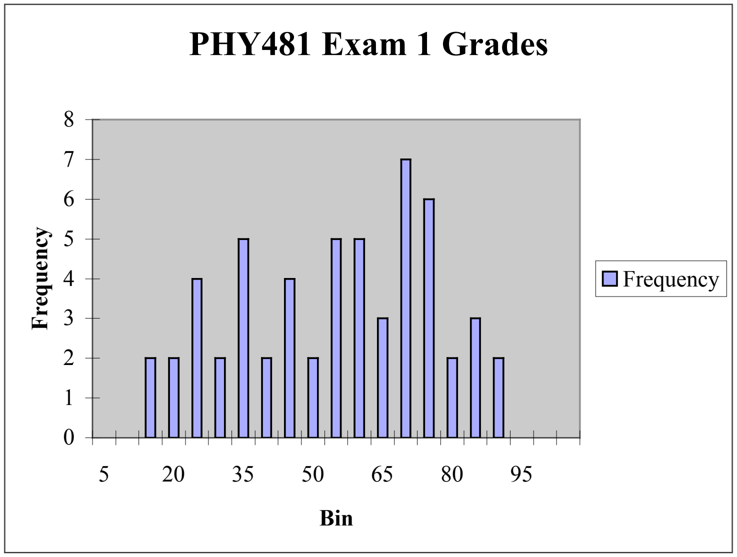
# PHY481: Electromagnetism

## Method of Images

## Rectangular, Spherical & Cylindrical Symmetry

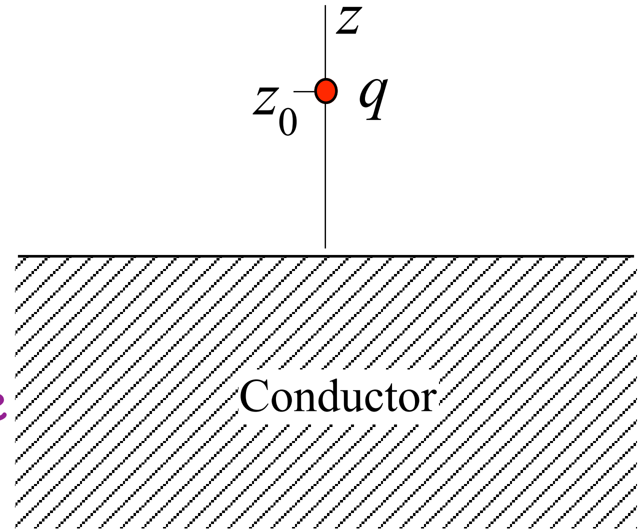


# What kind of problems need the image method?

V = 0 boundary condition

Charge $q$ above a grounded conductor.

What are the potential & field above, and surface charge density on, the conductor?

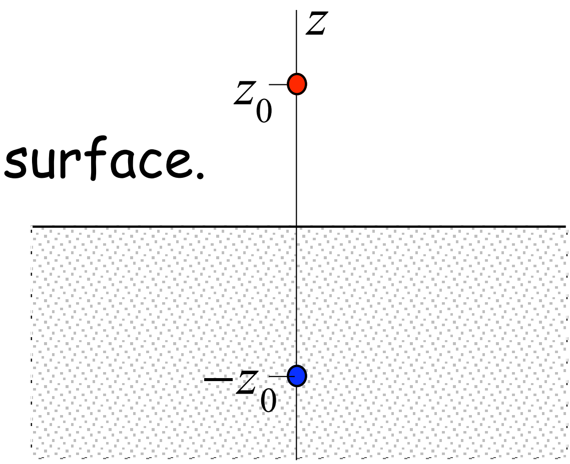



You must recognize that this is a problem where the symmetry implies a solution can be "easily" found via the

Method of Images

Temporarily forget about the conductor

Replace it by reflection of the space above its surface.




If reflected charge is -q, we have a dipole!

Dipole potential is zero on the midplane just like the V = 0 boundary condition above.

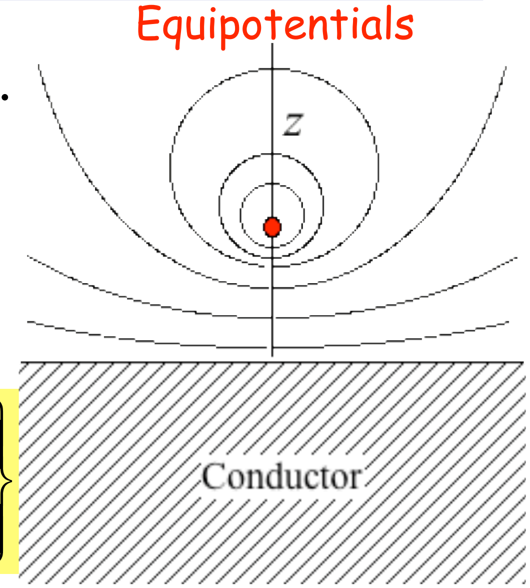
Dipole potential is a solution of Laplace's equation $\left(\mathbf{x} \neq \pm z_0 \hat{\mathbf{k}}\right)$

DONE ?

# Almost done

Dipole potential OK, but applies only above the plane.

$$V(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0}\left\{\left[x^2 + y^2 + (z - z_0)^2\right]^{(-1/2)} - \left[x^2 + y^2 + (z + z_0)^2\right]^{(-1/2)}\right\}$$

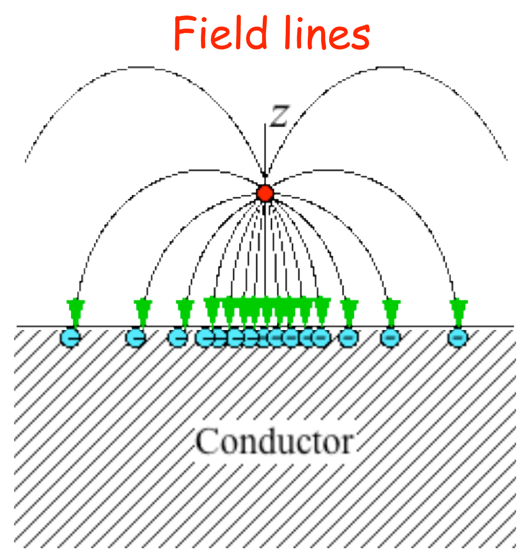



Field above the plane is the dipole field $\mathbf{E} = -\nabla V$

$$\mathbf{E}(x,y,z) = \frac{q}{4\pi\varepsilon_0}\left\{\frac{x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + (z - z_0)\hat{\mathbf{k}}}{\left[x^2 + y^2 + (z - z_0)^2\right]^{3/2}} - \frac{x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + (z + z_0)\hat{\mathbf{k}}}{\left[x^2 + y^2 + (z + z_0)^2\right]^{3/2}}\right\}$$

Field at the conductor's surface

$$\mathbf{E}(x,y,0) = \frac{-qz_0\,\hat{\mathbf{k}}}{2\pi\varepsilon_0\left[r^2 + z_0^2\right]^{3/2}} = E_n\,\hat{\mathbf{k}}$$

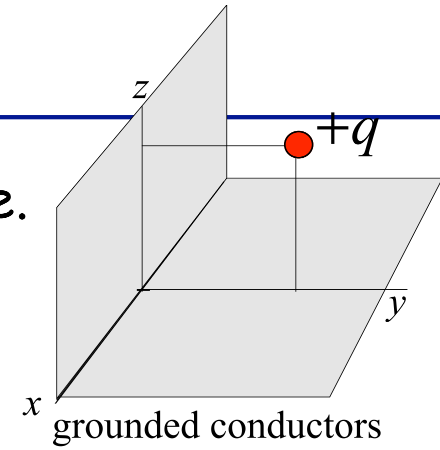



Conductor's surface charge density

$$\sigma(x,y) = \varepsilon_0 E_n(x,y) \qquad = \frac{-qz_0}{2\pi\left[r^2 + z_0^2\right]^{3/2}}$$

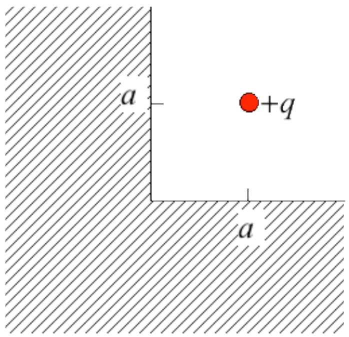
Now you are done!

# New image problem




Charge $q$ above a grounded conductors forming a vee.
What are the potential & field for $y,z > 0$, and
surface charge density on the conductors?

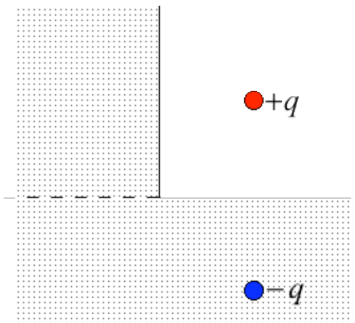
Problem to solve




instead

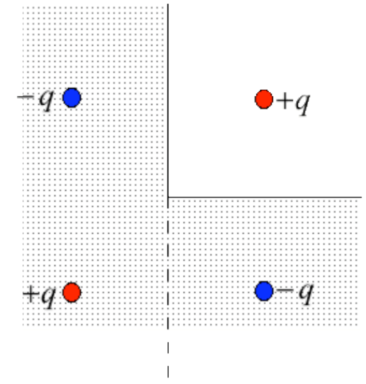
Get potential zero at z = 0

Reflect on horizontal axis




Get potential zero at y = 0

Reflect on vertical axis



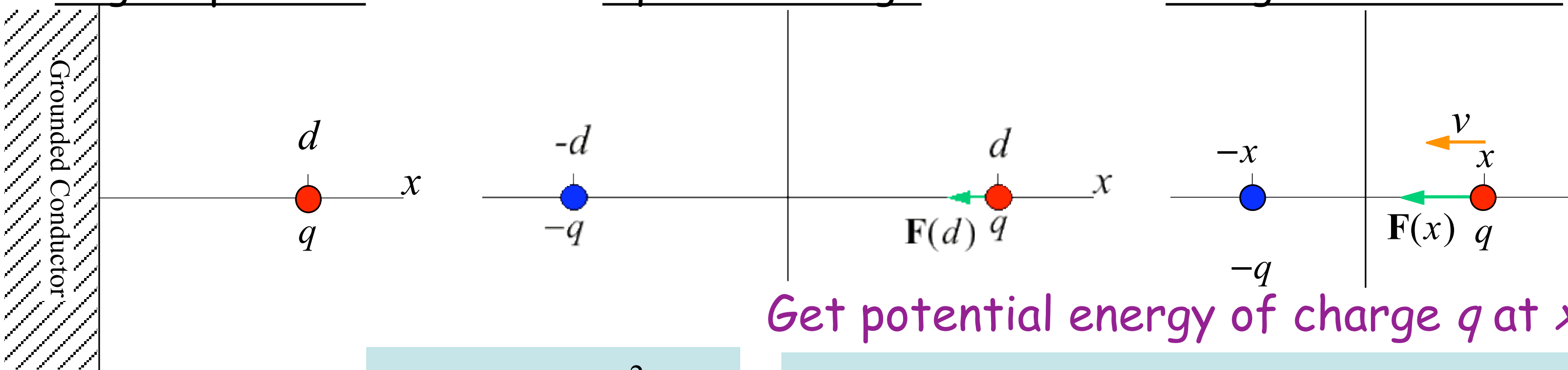

Total of +q and image charges is zero. --> +q will draw -q onto conductors
Potential in the notch will be that of opposing dipoles at y = ± a
Potential of opposing dipoles will tend to cancel far from the origin.
This leaves only a "quadrupole" potential in a multipole expansion .
Potential is "saddle shaped" at the corner , -q's at stirrups, +q's in front & back

# Practical image problem

Charge $q$, mass $m$ is released a distance $d$ from a grounded plane.

Negative charge drawn from ground. Positive charge hits plane at time $t$.

original problem — equivalent image — charges on the move




Get potential energy of charge $q$ at $x$

Use energy conservation

$$\mathbf{F}(x) = \frac{-q^2}{4\pi\varepsilon_0 (2x)^2}\,\mathbf{i}$$

$$U(x) = -\int_{\infty}^{x} \mathbf{F}\cdot\hat{\mathbf{i}}\,dx' = \frac{q^2}{4\pi\varepsilon_0}\int_{\infty}^{x}\frac{dx'}{(2x')^2} = \frac{-q^2}{16\pi\varepsilon_0 x}$$

Change in potential energy from $d$ to $x$

$$\Delta U(x) = U(x) - U(d) = \frac{-q^2}{16\pi\varepsilon_0}\left[\frac{1}{x} - \frac{1}{d}\right]$$

Energy conservation $\Delta KE = -\Delta U(x)$

$$\frac{1}{2}mv^2 = -\Delta U(x) = \frac{q^2}{16\pi\varepsilon_0 d}\left[\frac{d-x}{x}\right]$$

solve for v

$$\frac{dx}{dt} = v = \frac{q}{\sqrt{8\pi\varepsilon_0 md}}\sqrt{\frac{d-x}{x}}$$

integrate over $t'$ and $x'$

$$\int_0^t dt' = \frac{\sqrt{8\pi\varepsilon_0 md}}{q}\int_d^0\sqrt{\frac{x'}{d-x'}}\,dx'$$

http://integrals....

$$t = \frac{\sqrt{8\pi\varepsilon_0 md}}{q}\,\frac{\pi d}{2}$$

# Laplace's equation with spherical symmetry

Spherical coordinates

$$\nabla^2 V(r) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV(r)}{dr}\right) = 0$$

Guess a solution

$$V(r) = \frac{A}{r} + B$$

e.g., point charge

$$V = \frac{q}{4\pi\varepsilon_0 r}$$

Don't like to Guess? Determine a solution:

$$\text{Let } \frac{dV(r)}{dr} = f(r)$$

$$r^2\nabla^2 V = \frac{d}{dr}\left(r^2 f(r)\right) = 0$$

$$= 2rf(r) + r^2\frac{df(r)}{dr}$$

$$\frac{df(r)}{f(r)} = -2\frac{dr}{r}$$

$$f(r) = -Ar^{-2}$$

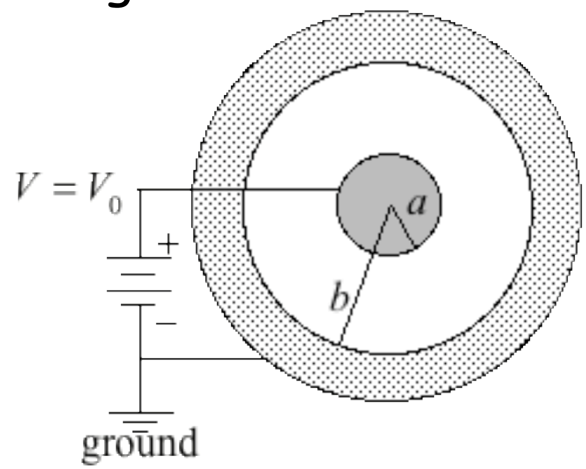
$$\frac{dV(r)}{dr} = -Ar^{-2}$$

General solution

$$V(r) = Ar^{-1} + B$$

# The spherical capacitor

Battery keeps an inner sphere at potential $V_0$ while outer sphere is grounded. What are potential, field and charge densities?




General solution

$$V(r) = Ar^{-1} + B$$

Boundary conditions

$$V(a) = V_0; \quad V(b) = 0$$

Apply boundary conditions to get A, B

$$V(a) = \frac{A}{a} + B = V_0 \qquad V(b) = \frac{A}{b} + B = 0$$

$$B = -\left(\frac{a}{b-a}\right)V_0 \qquad A = \left(\frac{ab}{b-a}\right)V_0$$

Complete the potential

$$V(r) = \frac{V_0 a}{r}\left(\frac{b-r}{b-a}\right)$$

Potential clearly satisfies boundary conditions at $r = a$ or $b$

Calculate the field

$$\mathbf{E} = -\nabla V(r) = \frac{V_0}{r^2}\left(\frac{ab}{b-a}\right)\hat{\mathbf{r}}$$

Charge densities:

$$\sigma_a = \varepsilon_0 E_n(a) = \varepsilon_0 V_0 \frac{b}{a}\frac{1}{b-a}$$

$$\sigma_b = \varepsilon_0 E_n(b) = -V_0 \frac{a}{b}\frac{1}{(b-a)}$$

Charges:

$$Q_a = \left(4\pi a^2\right)\sigma_a = 4\pi\varepsilon_0 V_0 \frac{ab}{b-a} = Q$$

$$Q_b = \left(4\pi b^2\right)\sigma_b = -4\pi\varepsilon_0 V_0 \frac{ab}{b-a} = -Q$$

# Laplace's equation with cylindrical symmetry

Cylindrical coordinates

$$\nabla^2 V(r) = \frac{1}{r}\frac{d}{dr}\left( r \frac{dV(r)}{dr} \right) = 0$$

Guess a solution

$$V(r) = A \ln\left(\frac{r}{r_0}\right) + B$$

e.g., Line charge

$$V = \frac{-\lambda}{2\pi\varepsilon_0} \ln\left(\frac{r}{a}\right)$$

Determine a solution

$$\text{Let } \frac{dV(r)}{dr} = f(r)$$

$$r\nabla^2 V = \frac{d}{dr}\left(rf(r)\right) = 0$$

$$= f(r) + r\frac{df(r)}{dr}$$

$$\frac{df(r)}{f(r)} = -\frac{dr}{r}$$

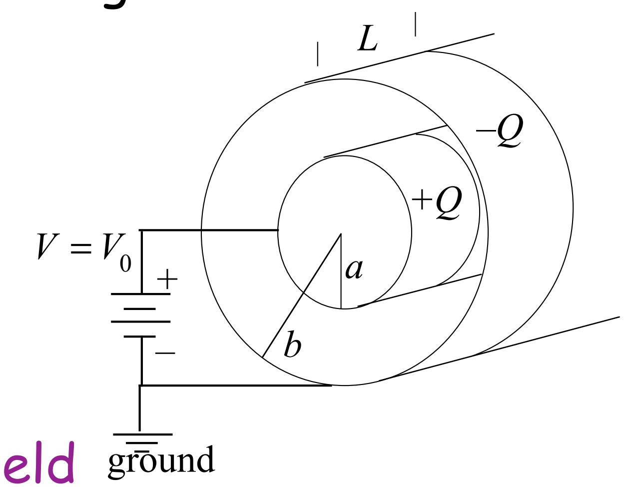
$$f(r) = Ar^{-1} = \frac{dV}{dr}$$

$$\frac{dV(r)}{dr} = Ar^{-1}$$

General solution $\quad V(r) = A\ln\left(r/r_0\right) + B$

# Cylindrical capacitor

Battery keeps an inner cylinder at potential $V_0$ while outer one is grounded. What are potential, field and charge densities?




General solution

$$V(r) = A\ln(r/r_0) + B$$

Boundary conditions

$$V(b) = 0; \quad V(a) = V_0$$

Apply boundary conditions to get A, B

$$0 = A\ln(b/r_0) + B \qquad V_0 = A\ln(a/r_0) + B$$

$$A = -V_0/\ln(b/a) \qquad B = +V_0\ln(b/r_0)/\ln(b/a)$$

Complete potential

$$V(r) = V_0\frac{\ln(b/r)}{\ln(b/a)}$$

boundary conditions clearly satisfied at $r = a$ or $b$

Calculate field

$$\mathbf{E} = -\nabla V(r) = \frac{V_0}{r\ln(b/a)}\hat{\mathbf{r}}$$

Charge densities

$$\sigma_a = \varepsilon_0 E_n(a) = \frac{\varepsilon_0 V_0}{a\ln(b/a)}$$

$$\sigma_b = \varepsilon_0 E_n(b) = -\frac{\varepsilon_0 V_0}{b\ln(b/a)}$$

Charge/unit L

$$\tilde{Q}_a = (2\pi a)\sigma_a = \frac{2\pi\varepsilon_0 V_0}{\ln(b/a)}$$

$$\tilde{Q}_b = (2\pi b)\sigma_b = -\frac{2\pi\varepsilon_0 V_0}{\ln(b/a)}$$

Capacitance/unit L

$$\tilde{C} = \frac{2\pi\varepsilon_0}{\ln(b/a)}$$

# Laplace's equation with angular dependence

Spherical coordinates (polar angle $\theta$)

$$\nabla^2 V(r,\theta) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) + \frac{1}{r\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right) = 0$$

Guess a solution (one of many)

$$V = Ar^{-1} + B + \frac{C\cos\theta}{r^2} + Dr\cos\theta$$

e.g., point dipole

$$V = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

or

linear V

$$V = E_0 z$$

Cylindrical coordinates (azimuth angle $\phi$)

$$\nabla^2 V(r,\phi) = \frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) + \frac{1}{r^2}\frac{d^2V}{d\phi^2} = 0$$

Guess a solution (one of many)

$$V = A\ln\left(\frac{r}{r_0}\right) + B + \frac{C\cos\phi}{r} + Dr\cos\phi$$

e.g., line dipole

$$V = \frac{E_0 a^2\cos\theta}{r}$$

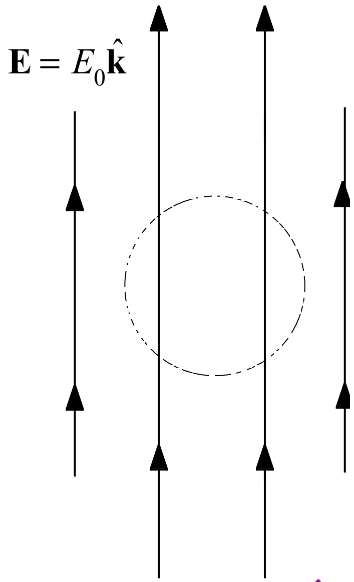
or

linear V

$$V = E_0 x$$

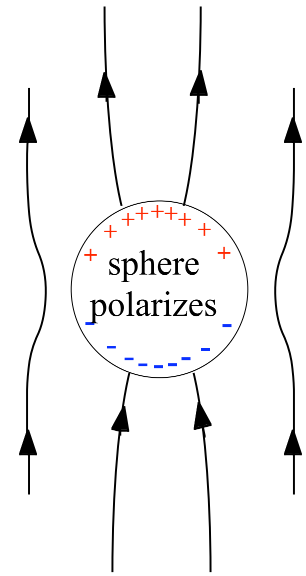
# Classic problem

Before




Grounded conducting sphere is inserted into a uniform field in z direction. What is the new field and charge density?

After




## General solution

$$V(r,\theta) = \frac{A}{r} + B + \frac{C\cos\theta}{r^2} + Dr\cos\theta$$

## Boundary conditions

$$Q = 0 \quad \Rightarrow \quad A = 0$$

$$V(a,\theta) = 0; \quad V(r,\theta)\big|_{r\to\infty} = -E_0 z$$

## Apply boundary conditions

$$B = 0 \quad \text{arbitrary}$$

$$V(r,\theta)\big|_{r\to\infty} = -E_0 z$$

$$D = -E_0 \ (z = r\cos\theta)$$

$$V(a,\theta) = 0 = \frac{C\cos\theta}{a^2} - E_0 a\cos\theta$$

$$C = E_0 a^3$$

## Complete solution

$$V(r,\theta) = E_0 r\cos\theta\left(\frac{a^3}{r^3} - 1\right)$$

## Field

$$\mathbf{E} = E_0\left(1 + \frac{2a^3}{r^3}\right)\cos\theta\,\hat{\mathbf{r}} + E_0\left(1 - \frac{a^3}{r^3}\right)\sin\theta\,\hat{\boldsymbol{\theta}}$$

## Charge density

$$\sigma(\theta) = 3\varepsilon_0 E_0\cos\theta$$