PHY481: Electromagnetism

Another image problem
Problems with specific symmetries
Solutions via “Separation of Variables”
1) Cartesian coordinates: Fourier Series
Spherical image problem

Charge $q_0$, a distance $z_0$ from the center of a grounded sphere. Force on $q_0$?

V must be zero at these points

$$\frac{q_0}{z_0 - R} = \frac{-q_1}{R - z_1}$$

$$\frac{q_0}{z_0 + R} = \frac{-q_1}{z_1 + R}$$

simultaneous eq.

Replace sphere with image charge $q_1$, a distance $z_1$ from the center.

$$q_1 = -q_0 \frac{R}{z_0}; \quad z_1 = \frac{R^2}{z_0}$$

Two charge potential

$$V(r, \theta) = \frac{1}{4\pi\varepsilon_0} \left[ \frac{q_0}{r_0} + \frac{q_1}{r_1} \right]$$

Field & PE

$$E = -\nabla V$$

$$W = U = -\int_{\infty}^{z_0} F_z \, dz$$
Summary: solving Laplace’s equation (1)

1) Grounded conductor and an external charge or dipole:
   Potential solved by the Method of Images
   Rectangular, Spherical, and Cylindrical symmetry

2) Potential between conductors: capacitors
   \[ Q = CV \]
   
   Parallel plate capacitors
   \[ V(z) = Az + B \]

   Spherical capacitors
   \[ V(r) = Ar^{-1} + B \]

   Cylindrical capacitors
   \[ V(r) = A \ln\left(\frac{r}{r_0}\right) + B \]

3) Potentials for conductors in external fields

   Spherical symmetry
   \[ V(r, \theta) = Ar^{-1} + B + \frac{C \cos \theta}{r^2} + Dr \cos \theta \]

   Cylindrical symmetry
   \[ V(r, \phi) = A \ln\left(\frac{r}{r_0}\right) + B + \frac{C \cos \phi}{r} + Dr \cos \phi \]
Solving Laplace’s equation (2)

In general, a solution of Laplace’s equation subject to boundary conditions requires an infinite series of functions.

**Cartesian coordinate boundary conditions: Fourier series**

Any periodic function with period $2a$, can be expanded in a “Fourier Series” of sine and cosine functions, with $n$ a positive integer

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi x}{a} + b_n \sin \frac{n\pi x}{a} \right]$$

“Orthogonality” of the trigonometric functions

$$\frac{1}{a} \int_{-a}^{a} \cos \frac{n\pi x}{a} \cos \frac{m\pi x}{a} \, dx = \delta_{nm}$$

$$\frac{1}{a} \int_{-a}^{a} \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a} \, dx = \delta_{nm}$$

The coefficients $a_n$ and $b_n$ are determined by doing these integrals

$$a_0 = \frac{1}{2a} \int_{-a}^{a} f(x) \, dx$$

$$a_m = \frac{1}{a} \int_{-a}^{a} f(x) \cos \frac{m\pi x}{a} \, dx$$

$$b_m = \frac{1}{a} \int_{-a}^{a} f(x) \sin \frac{m\pi x}{a} \, dx$$
Odd or even functions

Odd functions select these

Even functions select these

Odd n terms

Even n terms
Qualitative expansion of $f(x) = V_0$

Fourier series expansion

$f(x) = V_0$  \((-a/2 < x < +a/2)\)

Extend $f(x)$ to make a periodic function

Note similarity to cosine function

Prepare a fraction of the next odd n cosine

Add to the first. Getting closer!

Add a few more terms. Pretty good match!

$\frac{-4}{3\pi}V_0 \cos\left(\frac{3\pi x}{a}\right)$
Coefficients in a Fourier Series

Find Fourier series expansion of \( f(x) = V_0 \)

\[
f(x) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi x}{a} + b_n \sin \frac{n\pi x}{a} \right]
\]

- \( b_m = \frac{1}{a} \int_{-a}^{a} f(x) \sin \frac{m\pi x}{a} \, dx = 0 \)
- function \( f(x) \) is even so no sin terms

\[
a_m = \frac{1}{a} \int_{-a}^{a} f(x) \cos \frac{m\pi x}{a} \, dx
= \frac{1}{m\pi} \left[ -V_0 \sin \frac{m\pi x}{a} \right]_{-a}^{-a/2} + V_0 \sin \frac{m\pi x}{a} \right]_{-a/2}^{+a/2} - V_0 \sin \frac{m\pi x}{a} \right]_{+a/2}^{+a}
\]

- \( a_m = \begin{cases} 
\frac{4V_0}{m\pi} (-1)^n, & m = 2n + 1 \text{ (odd)} \\
0, & m = 2, 4, \ldots
\end{cases} \)

\[
f(x) = \frac{4V_0}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos \left( \frac{(2n+1)\pi x}{a} \right)
\]
Solution by “Separation of Variables”

Laplace’s equation for an electric potential in Cartesian coordinates

\[ \nabla^2 V(x) = \frac{\partial^2 V(x)}{\partial x^2} + \frac{\partial^2 V(x)}{\partial y^2} + \frac{\partial^2 V(x)}{\partial z^2} = 0 \]

Solutions are often of the “separable” type, or linear combinations of such solutions:

\[ V(x) = X(x)Y(y)Z(z) \]

Partial derivatives in Laplace’s equation become total derivatives

\[ \nabla^2 V(x) = \frac{d^2 X(x)}{dx^2} YZ + \frac{d^2 Y(y)}{dy^2} XZ + \frac{d^2 Z(z)}{dz^2} XY = 0 \]

Divide by \( V \)

\[ \frac{\nabla^2 V(x)}{V(x)} = \frac{1}{X} \frac{d^2 X(x)}{dx^2} + \frac{1}{Y} \frac{d^2 Y(y)}{dy^2} + \frac{1}{Z} \frac{d^2 Z(z)}{dz^2} = 0 \]
Coordinate independence in separable solutions

Cartesian coordinate separable solutions

\[ V(\mathbf{x}) = X(x)Y(y)Z(z) \]

Translational independence \( x \rightarrow x' \)

\[ V(\mathbf{x}') = X(x')Y(y)Z(z) \]

\[ \nabla^2 V / V = \frac{1}{X(x')} \frac{d^2 X(x')}{dx'^2} + \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} + \frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} = 0 \]

Values of the last two terms (\( Y \) and \( Z \) terms) have not changed. Yet the sum is still ZERO. Each of the 3 terms must be a constant.

\[ \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = k_1^2 \]
\[ \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = k_2^2 \]
\[ \frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} = k_3^2 \]

Solution to Laplace’s equation must have

\[ k_1^2 + k_2^2 + k_3^2 = 0 \]
**Range of solutions**

\[ k_1^2 + k_2^2 + k_3^2 = 0 \]

\[
\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = k_1^2; \quad \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = k_2^2; \quad \frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} = k_3^2
\]

If \( k_1 = 0 \) (simple field) function \( X \) satisfies

\[
1 \frac{d^2 X(x)}{X(x) dx^2} = 0
\]

solution: \( X(x) = ax + b \) \hspace{1cm} Constants \( a \) & \( b \) determined by the boundary conditions

If \( k_1^2 > 0 \), function \( X(x) \) satisfies

\[
\frac{d^2 X(x)}{dx^2} = k_1^2 X(x)
\]

solution: \( X(x) = Ae^{k_1 x} + Be^{-k_1 x} \) \hspace{1cm} etc., for \( Y \) & \( Z \)

If \( k_1^2 < 0 \), then \( k \rightarrow ik' \) \( k' \) real \( > 0 \), and \( X \) satisfies

\[
\frac{d^2 X(x)}{dx^2} = -k_1'^2 X(x)
\]

solution: \( X(x) = Ae^{ik_1' x} + Be^{-ik_1' x} \) \hspace{1cm} etc., for \( Y \) & \( Z \)
Character of solutions

$k_{1}^{2} > 0$

\[ X(x) = Ae^{k_{1}x} + Be^{-k_{1}x} \]

Allowable values of \( A, B, \) and \( k_{1} \) determined by symmetries and boundary conditions

If region unbounded +x

\[ A = 0, \quad X(x) = Be^{-k_{1}x} \]

If region unbounded -x

\[ B = 0, \quad X(x) = Ae^{k_{1}x} \]

If region bounded + & -

\[ X(x) = A' \cosh k_{1}x + B' \sinh k_{1}x \]

Even symmetry

\[ \cosh k_{1}x = \left( \frac{e^{k_{1}x} + e^{-k_{1}x}}{2} \right); \quad \sinh k_{1}x = \left( \frac{e^{k_{1}x} - e^{-k_{1}x}}{2} \right) \]

Odd symmetry

\[ Suggests \ using \ Fourier \ series \ for \ A, B, k_{1} \]

$k_{1}^{2} < 0$

\[ X(x) = Ae^{i k_{1}'x} + Be^{-i k_{1}'x} \]

\[ X(x) = A' \cos k_{1}'x + B' \sin k_{1}'x \]

Even symmetry

\[ \cos k_{1}x = \left( \frac{e^{ik_{1}x} + e^{-ik_{1}x}}{2} \right); \quad \sin k_{1}x = \left( \frac{e^{ik_{1}x} - e^{-ik_{1}x}}{2i} \right) \]

Odd symmetry
**Example: Long narrow channel**

Large top and bottom grounded plates, potential $V_0$ on the end plate
Find potential everywhere between the plates.

$L(x, y) = X(x)Y(y)$

problems without $z$ dependence
all start the same way

Laplace's equation

Each term must be a constant

Even $y$ symmetry

First determine $Y$

$Y(y) = A \cos ky + B \sin ky$

$Y(y) = A \cos ky \quad B = 0$

$V = 0$ on top and bottom plate

$Y(a/2) = \cos(ka/2) = 0$

$k = (2n + 1) \frac{\pi}{a}; \quad n = 0, 1, 2, \ldots$

Solution with coefficients

$Y(y) = \sum_{n=0}^{\infty} A_n \cos \left[ (2n + 1) \frac{\pi y}{a} \right]$
Example (cont’d)

Determine the $X$ function

$$\frac{d^2 X(x)}{dx^2} = +k^2 X(x)$$

General solution

$$X(x) = A e^{kx} + B e^{-kx}$$

Finite at large $+x$

$$X(x \to \infty) = 0, \ A = 0$$

$$X(x) = B e^{-kx}$$

From $Y$ solution

$$k = (2n+1) \frac{\pi}{a}; \ n = 0, 1, 2, \ldots$$

$$X(x) = B e^{-(2n+1)\pi x/a}$$

$V = XY$ and determine the coefficients

$$C_n = A_n B$$

$$V(x, y) = X(x)Y(y) = \sum_{n=0}^{\infty} C_n \cos \left( \frac{(2n+1)\pi y}{a} \right) e^{-(2n+1)\pi x/a}$$

On left boundary, Potential is $V_0$

$$V(0, y) = V_0 = \sum_{n=0}^{\infty} C_n \cos \left( \frac{(2n+1)\pi y}{a} \right)$$

Fourier tells us we get $C_n$ this way

$$C_n = \frac{1}{a} \int_{-a}^{a} V_0 \cos \left( \frac{(2n+1)\pi y}{a} \right) \frac{dx}{a} = \frac{4V_0 (-1)^n}{(2n+1)\pi}$$

Finally a solution

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)\pi y}{a} e^{-(2n+1)\pi x/a}$$