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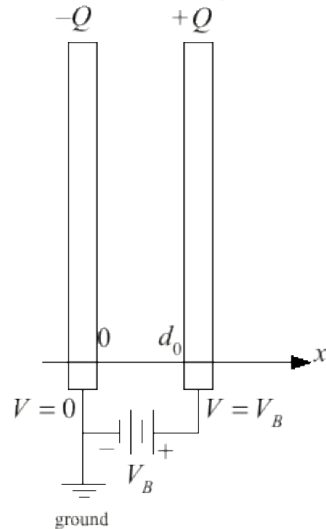
# PHY481: Electromagnetism

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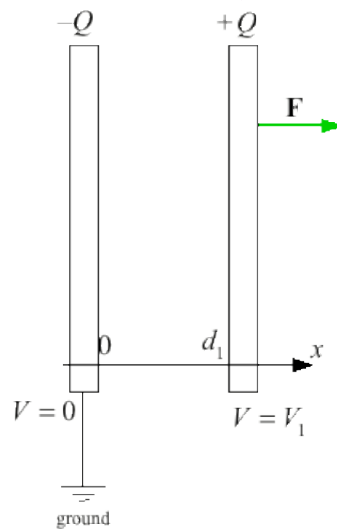
HW4

## 4.3

Plates charged



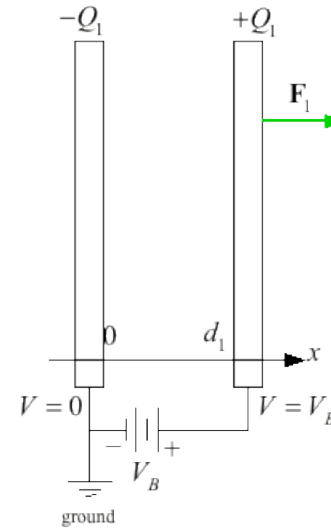
No battery



Charge remains at  $Q$   
Potential changes to  $V_1$

Spread plates

With battery



Charge changes to  $Q_1$   
Potential remains at  $V_B$

$$Q = CV_B = \epsilon_0 AV_B / d_0$$

$$Q = C_1 V_1 = \epsilon_0 AV_1 / d_1$$

$$Q_1 = C_1 V_B = \epsilon_0 AV_B / d_1$$

Stored energy

$$\mathcal{E} = \frac{1}{2} CV_B^2 = \frac{1}{2} \frac{\epsilon_0 A}{d_0} V_B^2$$

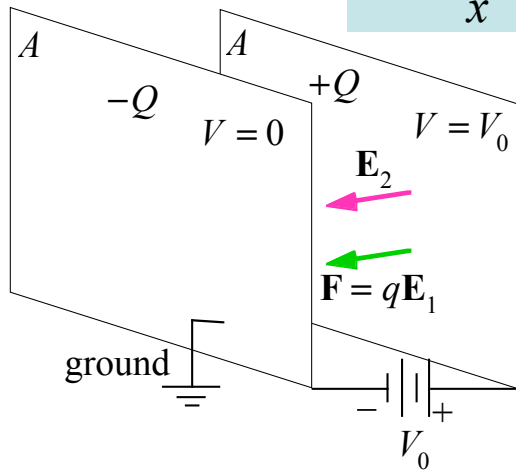
$$\mathcal{E}_1 = \frac{1}{2} C_1 V_1^2 = \frac{d_1}{d_0} \mathcal{E} \quad \uparrow$$

$$\mathcal{E}' = \frac{1}{2} C_1 V_B^2 = \frac{d_0}{d_1} \mathcal{E} \quad \downarrow ?$$

# 4.3

$$Q = CV$$

$$C = \frac{\epsilon_0 A}{x}$$



Field between the plates is the sum of the field of **both plates**

Force on +Q generated by field of the -Q charge on **one plate**

Battery disconnected

$$\mathbf{E}_2 = \sigma / \epsilon_0$$

$$\mathbf{E}_1 = \sigma / 2\epsilon_0$$

Battery connected

$$Q = Q_0$$

$$V = V_0$$

$$\mathbf{F} = q\mathbf{E}_1 = Q \frac{\sigma}{2\epsilon_0} =$$

$$\frac{Q_0^2}{2\epsilon_0 A} = \frac{\epsilon_0 A V_0^2}{2x_0^2}$$

$$\frac{\epsilon_0 A V_0^2}{2x^2}$$

Work done to pull plates apart by  $\Delta x$

$$W = - \int_{x_0}^{x_0 + \Delta x} \mathbf{F} \cdot \hat{\mathbf{i}} dx =$$

$$\frac{\epsilon_0 A V_0^2}{2} \frac{\Delta x}{x_0^2}$$

$$\frac{\epsilon_0 A V_0^2}{2} \frac{\Delta x}{x_0(x_0 + \Delta x)}$$

Cap. energy storage

$$U = \frac{Q^2}{2C}$$

Energy change

$$\Delta U = \frac{Q^2}{2C} - \frac{Q_0^2}{2C_0} =$$

$$\frac{\epsilon_0 A V_0^2}{2} \frac{\Delta x}{x_0^2}$$

negative

$$-\frac{\epsilon_0 A V_0^2}{2} \frac{\Delta x}{x_0(x_0 + \Delta x)}$$

Energy stored by Battery

$$\Delta \mathcal{E} = \Delta Q V_0 =$$

$$0$$

$$\epsilon_0 A V_0^2 \frac{\Delta x}{x_0(x_0 + \Delta x)}$$

## 4.4

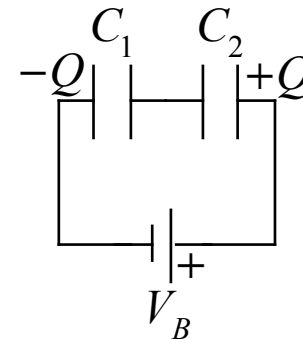
### Capacitors in Series

$$V_B = V_1 + V_2; \quad Q = Q_1 = Q_2$$

$$\frac{Q}{C} = \frac{Q}{C_1} + \frac{Q}{C_2}$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$\frac{1}{C} = \sum_{i=1}^n \frac{1}{C_i}$$



### Capacitors in Parallel

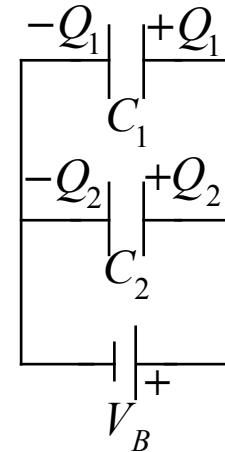
$$Q = Q_1 + Q_2; \quad V_B = V_1 = V_2;$$

$$= C_1 V_B + C_2 V_B$$

$$C V_B = (C_1 + C_2) V_B$$

$$C = C_1 + C_2$$

$$C = \sum_{i=1}^n C_i$$



## 4.6

Potential of charge over conducting plane, via MoI is a discrete dipole potential

$$V(\rho, z) = \frac{q}{4\pi\epsilon_0} \left[ \frac{1}{\sqrt{(\rho^2 + (z - z_0)^2)}} - \frac{1}{\sqrt{(\rho^2 + (z + z_0)^2)}} \right]$$

Radius on the mid-plane

$$\rho^2 = x^2 + y^2$$

Electric field on the mid-plane

$$E_z(\rho, 0) = - \left. \frac{\partial V}{\partial z} \right|_{z=0} = - \frac{q}{2\pi\epsilon_0} \frac{z_0}{(\rho^2 + z_0^2)^{3/2}}$$

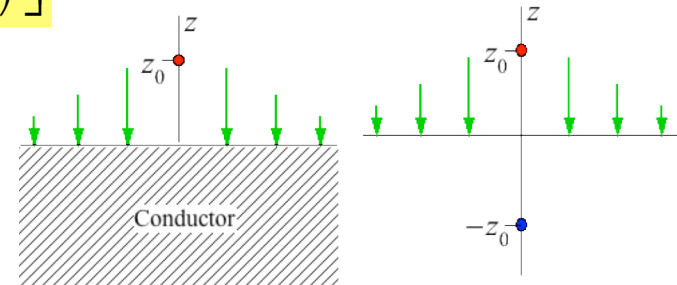
Charge density on the conductor surface

$$\sigma(\rho) = \epsilon_0 E_n(\rho, 0) = - \frac{q}{2\pi} \frac{z_0}{(\rho^2 + z_0^2)^{3/2}}$$

Total charge on the on conductor surface

$$Q = 2\pi \int_0^\infty \sigma(\rho) \rho d\rho = -qz_0 \int_0^\infty \frac{\rho d\rho}{(\rho^2 + z_0^2)^{3/2}}$$

$$= -qz_0 \left[ \frac{-1}{(\rho^2 + z_0^2)^{1/2}} \right]_0^\infty = -q \quad \left( \begin{array}{c} \text{as} \\ \text{expected} \end{array} \right)$$



Radius enclosing half the surface charge

$$\frac{-q}{2} = 2\pi \int_0^r \sigma(\rho) \rho d\rho = qz_0 \left[ \frac{1}{(\rho^2 + z_0^2)^{1/2}} \right]_0^r$$

$$\frac{1}{2} = \frac{1}{(x^2 + 1)^{1/2}}; \quad x = r/z_0$$

$$x = r/z_0; \quad x = \sqrt{3}; \quad r = \sqrt{3}z_0$$

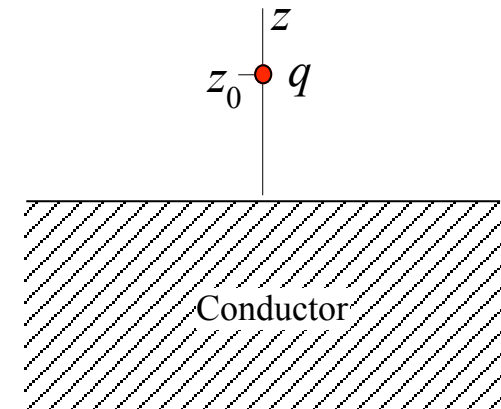
## 4.7

Force on  $+q$  from negative charge  
drawn onto the conductor

Force on  $+q$  from negative charge  $-q$  in the image.

$$F = \frac{q^2}{4\pi\epsilon_0 r^2} = \frac{q^2}{4\pi\epsilon_0 (2z_0)^2} = \frac{q^2}{16\pi\epsilon_0 z_0^2}$$

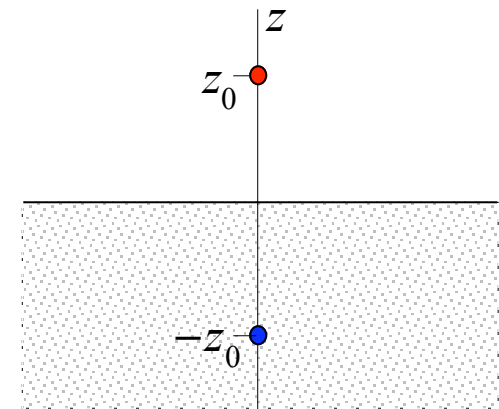
$$\mathbf{F} = -\frac{q^2}{16\pi\epsilon_0 z_0^2} \hat{\mathbf{k}}$$



Energy in field is work done to move  $+q$  to the point  $z_0$

$$U = W = -\int_{\infty}^{z_0} \mathbf{F} \cdot \hat{\mathbf{k}} dr = \frac{q^2}{16\pi\epsilon_0} \int_{\infty}^{z_0} \frac{dr}{r^2} = \frac{q^2}{16\pi\epsilon_0} \left[ -\frac{1}{r} \right]_{\infty}^{z_0}$$

$$= \frac{-q^2}{16\pi\epsilon_0 z_0}$$



Another way to get field energy. Use delta function charge distribution

$$U = \frac{1}{2} \int \rho(\mathbf{x}) V(\mathbf{x}) d^3x = \frac{1}{2} \int q \delta^3(\mathbf{x} - z_0 \hat{\mathbf{k}}) \frac{-q}{4\pi\epsilon_0 |\mathbf{x} + z_0 \hat{\mathbf{k}}|} d^3x = \frac{-q^2}{8\pi\epsilon_0 |z_0 \hat{\mathbf{k}} + z_0 \hat{\mathbf{k}}|} = \frac{-q^2}{16\pi\epsilon_0 z_0}$$

## 4.10a

Find potential of an upward facing dipole above a grounded conductor

Potential via M.o.I is two dipoles (SAME DIRECTION)

Single dipole potential

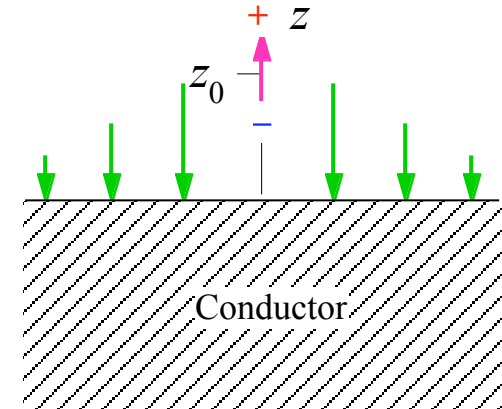
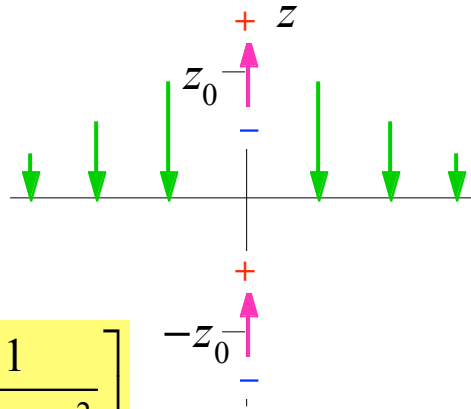
$$V_d(r, \theta) = \frac{p \cos \theta}{4\pi\epsilon_0 r^2}$$

Double dipole potential

$$V_{2d}(z, 0) = \frac{p}{4\pi\epsilon_0} \left[ \frac{1}{(z - z_0)^2} + \frac{1}{(z + z_0)^2} \right]$$

$$V_{2d}(z, 0) = \frac{p}{2\pi\epsilon_0 z^2}, \quad z \gg z_0$$

$$\mathbf{E} = -\Delta V = \frac{p}{\pi\epsilon_0 z^3} \hat{\mathbf{k}}$$



## 4.14

Use Gauss's Law to determine the field and potential between shells.

$$\int_S \mathbf{E} \cdot d\mathbf{A} = \frac{q_{enc}}{\epsilon_0} \rightarrow \mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}} \quad V(r) = -\int_b^r \mathbf{E} \cdot \hat{\mathbf{r}} dr = \frac{Q}{4\pi\epsilon_0} \left[ \frac{1}{r} - \frac{1}{b} \right]$$

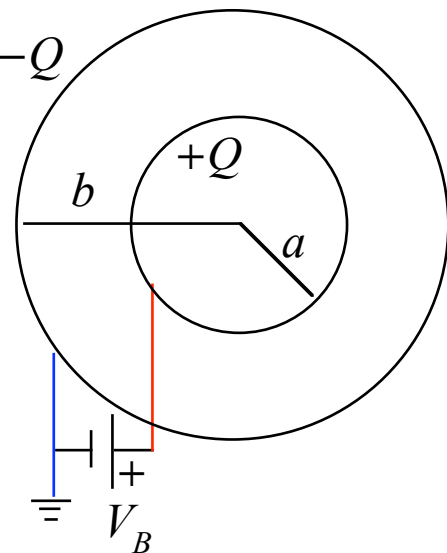
Determine the field energy

$$U = \frac{\epsilon_0}{2} \int E^2 d^3x = \frac{Q^2}{8\pi\epsilon_0} \int_a^b \frac{1}{r^4} r^2 dr = \frac{Q^2}{8\pi\epsilon_0} \left( -\frac{1}{r} \right)_a^b = \frac{Q^2}{8\pi\epsilon_0} \left( \frac{b-a}{ab} \right) - Q$$

Determine the capacitance

$$U = \frac{1}{2} C V_B^2; \quad V_B = V(a) = \frac{Q}{4\pi\epsilon_0} \left[ \frac{1}{a} - \frac{1}{b} \right]$$

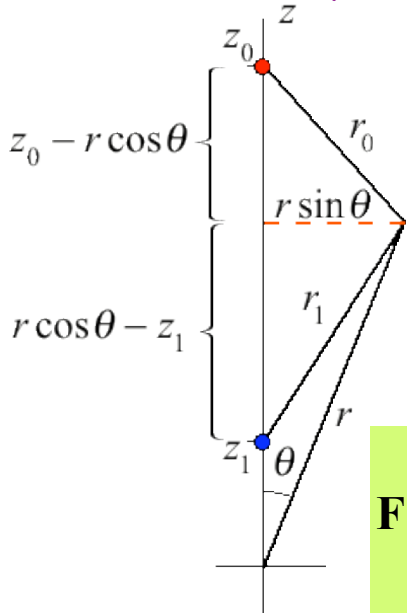
$$C = \frac{2U}{V_B^2} = \frac{\frac{Q^2}{4\pi\epsilon_0} \left( \frac{b-a}{ab} \right)}{\left[ \frac{Q}{4\pi\epsilon_0} \left( \frac{b-a}{ab} \right) \right]^2} = 4\pi\epsilon_0 \frac{ab}{b-a}$$



# 4.15

Potential via M.o.I uses  $q_2$  at  $z_2$  Determine them using points a,b

Geometry



$$V_a = 0$$

$$\frac{q_0}{z_0 - R} = \frac{-q_1}{R - z_1}$$

Cross multiply

$$q_0(R - z_1) = -q_1(z_0 - R)$$

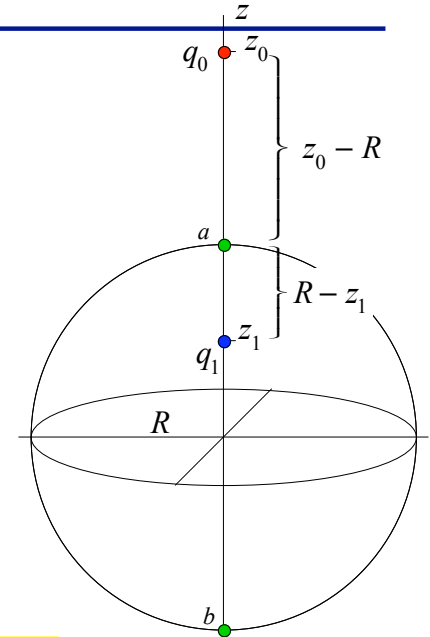
$$q_0(z_1 + R) = -q_1(z_0 + R)$$

$$V_b = 0$$

$$\frac{q_0}{z_0 + R} = \frac{-q_1}{z_1 + R}$$

Image

$$q_1 = -q_0 \frac{R}{z_0}; \quad z_1 = \frac{R^2}{z_0}$$



$$\mathbf{F} = \frac{-q_0^2 R z_0}{4\pi\epsilon_0 (z_0^2 - R^2)^2} \hat{\mathbf{k}}$$

$$U = -\int_{\infty}^{z_0} F_z dz = -\frac{q_0^2 R}{4\pi\epsilon_0} \int_{\infty}^{z_0} \frac{z dz}{(z^2 - R^2)^2}$$

$$= -\frac{q_0^2 R}{8\pi\epsilon_0} \frac{1}{(z_0^2 - R^2)}$$

$$r_0 = \sqrt{r^2 + z_0^2 - 2r z_0 \cos \theta}$$

$$r_1 = \sqrt{z_0^2 r^2 + R^4 - 2r z_0 R^2 \cos \theta} / z_0$$

$$E_r(r, \theta) = -\frac{\partial V}{\partial r} = \frac{q}{4\pi\epsilon_0} \left[ \frac{r - z_0 \cos \theta}{r_0^3} + \frac{R(z_0^2 r - z_0 R^2 \cos \theta)}{r_1^3} \right]$$

$$V(r, \theta) = \frac{1}{4\pi\epsilon_0} \left[ \frac{q_0}{r_0} + \frac{q_1}{r_1} \right]$$

$$E_\theta(r, \theta) = -\frac{\partial V}{r \partial \theta} = \frac{q z_0 \sin \theta}{4\pi\epsilon_0} \left[ \frac{1}{r_0^3} + \frac{R^3 / z_0^3}{r_1^3} \right]$$

## 4.16

Charge  $q_0$  above a hemispherical "blister" on a flat grounded plane.

Use Method of Images to determine potential above the plane

1) Get  $V = 0$  on blister surface.

See problem 4.15 for image of  $q_0$  :

$$q_1 = -q_0 \frac{R_0}{z_0}; \quad z_1 = \frac{R_0^2}{z_0}$$

2) Get  $V = 0$  on plane.

$q_2$  is image of  $q_1$  :

$$q_2 = q_0 \frac{R_0}{z_0}; \quad z_2 = -\frac{R_0^2}{z_0}$$

$q_3$  is image of  $q_0$  :

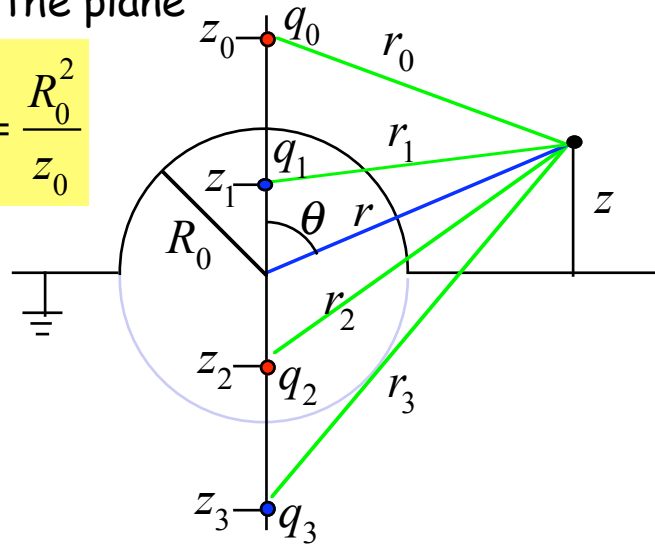
$$q_3 = -q_0; \quad z_3 = -z_0$$

2)  $V(r, \theta)$  is sum of the point potentials.

$$V(r, \theta) = \frac{1}{4\pi\epsilon_0} \sum_{i=0}^3 \left[ \frac{q_i}{r_i} \right]$$

$$r_i = |\mathbf{x} - \mathbf{x}_i| = \left[ r^2 + z_i^2 - 2rz_i \cos \theta \right]^{1/2}$$

3) Answer given suggests that the easiest way to find the hemisphere's charge is to subtract the charge on the plane from the total charge:  $q_1 + q_2 + q_3 = -q_0$



On the plane the normal component

of the field is in the direction  $\hat{\theta}|_{\theta=\pi/2} = -\hat{\mathbf{k}}$

$$E_z(r, \pi/2) = -E_\theta(r, \pi/2)$$

$$E_\theta = -\frac{1}{r} \sum_{i=0}^3 \frac{\partial V_i}{\partial \theta} = \frac{1}{4\pi\epsilon_0} \sum_{i=0}^3 \frac{q_i z_i \sin \theta}{\left[ r^2 + z_i^2 - 2rz_i \cos \theta \right]^{3/2}}$$

## 4.16 (cont)

Charge density on the plane

$$E_z(r, \theta) = -E_\theta(r, \theta)$$

$$\text{when } \sin(\pi/2) = 1, \cos(\pi/2) = 0$$

$$\sigma(r) = \varepsilon_0 E_z(r, \theta) \big|_{\theta=\pi/2} = \frac{-1}{4\pi} \sum_{i=0}^3 \frac{q_i z_i}{[r^2 + z_i^2]^{3/2}}$$

Total charge on the plane

$$Q_p = \int_0^{2\pi} d\phi \int_{R_0}^{\infty} \sigma(r) r dr = \frac{-1}{2} \sum_{i=0}^3 q_i z_i \int_{R_0}^{\infty} [r^2 + z_i^2]^{-3/2} r dr = \frac{-1}{2} \sum_{i=0}^3 q_i z_i (R_0^2 + z_i^2)^{-1/2}$$

Image charges:

$$q_1 = -q_0 \frac{R_0}{z_0}; \quad z_1 = \frac{R_0^2}{z_0}$$

$$q_2 = q_0 \frac{R_0}{z_0}; \quad z_2 = -\frac{R_0^2}{z_0}$$

$$q_3 = -q_0; \quad z_3 = -z_0$$

$$\begin{aligned} Q_p &= \frac{-1}{2} \left[ 2q_0 z_0 (R_0^2 + z_0^2)^{-1/2} - 2q_0 \left( R_0^3 / z_0^2 \right) (R_0^2 + R_0^4 / z_0^2)^{-1/2} \right] \\ &= q_0 (R_0^2 - z_0^2) / [z_0 (R_0^2 + z_0^2)^{1/2}] \end{aligned}$$

Using  $q_1 + q_2 + q_3 = -q_0$

$$f_{\text{hemi}} = [(-q_0) - Q_p] / (-q_0) = 1 - (z_0^2 - R_0^2) / [z_0 (R_0^2 + z_0^2)^{1/2}]$$

## 4.18

### Two spheres charged by a battery

Because the two spheres are very far apart the potentials of each sphere are independent.

Reference each potential with respect to  $V = 0$  at infinity

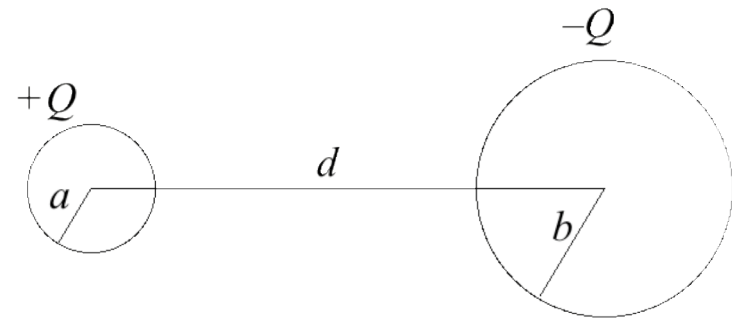
Comparing the potentials of each sphere with respect to  $V = 0$  at infinity

$$V(r) = \frac{q}{4\pi\epsilon_0 r}$$

$$\begin{aligned} V_0 = V(b) - V(a) &= \frac{Q}{4\pi\epsilon_0 b} - \frac{-Q}{4\pi\epsilon_0 a} \\ &= \frac{Q(a+b)}{4\pi\epsilon_0 ab} \end{aligned}$$

$$C = \frac{Q}{V_0} = 4\pi\epsilon_0 \frac{ab}{(a+b)}$$

Charged by a battery, each must have the same magnitude of charge



## 4.20

See lecture 16 for derivation of charge density on a **grounded sphere in an external field**

New boundary conditions

$$Q = Q_0 \Rightarrow A \neq 0$$

$$V'(a, \theta) = \frac{Q_0}{4\pi\epsilon_0 a}$$

New term

$$V'(r, \theta) = \frac{A}{r} + \frac{E_0 a^3 \cos \theta}{r^2} - E_0 r \cos \theta$$

$$V'(a, \theta) = \frac{A}{a} + E_0 a \cos \theta - E_0 a \cos \theta = \frac{Q_0}{4\pi\epsilon_0 a}$$

$$A = \frac{Q_0}{4\pi\epsilon_0}$$

$$E'_r(r, \theta) = -\frac{\partial V(r, \theta)}{\partial r} = \frac{Q_0}{4\pi\epsilon_0 r^2} + \frac{2E_0 a^3 \cos \theta}{r^3} + E_0 \cos \theta$$

$$E'_r(a, \theta) = \frac{Q_0}{4\pi\epsilon_0 a^2} + 3E_0 \cos \theta$$

$$\sigma'(\theta) = \epsilon_0 E'_r(a, \theta) = \frac{Q_0}{4\pi a^2} + 3\epsilon_0 E_0 \cos \theta$$

$$\sigma'(\theta) > 0; \cos \theta = -1$$

$$\frac{Q_0}{4\pi a^2} = 3\epsilon_0 E_0$$

$$Q_0 = 12\pi a^2 \epsilon_0 E_0$$

General potential spherical coordinates in simple situations

$$V(r, \theta) = \frac{A}{r} + B + \frac{C \cos \theta}{r^2} + Dr \cos \theta$$

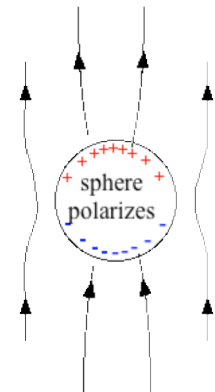
Grounded sphere in external field

$$Q = 0 \Rightarrow A = 0$$

$$V(a, \theta) = 0; \quad V(r, \theta)|_{r \rightarrow \infty} = -E_0 z$$

$$\sigma(\theta) = 3\epsilon_0 E_0 \cos \theta$$

Add positive charge  $Q_0$  here



## 4.22

### Line charge above a grounded conducting plane

Single line  
charge potential

$$V(r) = \frac{-\lambda \ln r}{2\pi\epsilon_0}$$

Double line  
charge potential

$$V(r) = \frac{-\lambda \ln(r_1/r_2)}{2\pi\epsilon_0}$$

Electric  
field

$$E_y = -(\nabla V)_y = \frac{\lambda}{2\pi\epsilon_0} \left[ \frac{y - y_0}{r^2 + y_0^2 - 2yy_0} - \frac{y + y_0}{r^2 + y_0^2 + 2yy_0} \right]$$

Field on  
mid-plane

$$E_y(x) = \frac{-\lambda y_0}{\pi\epsilon_0(x^2 + y_0^2)}$$

Charge density  
on conductor

$$\sigma(x) = \epsilon_0 E_y(x) = \frac{-\lambda y_0}{\pi(x^2 + y_0^2)}$$

Linear charge density  
on conductor

$$\lambda_z = \int_{-\infty}^{\infty} \sigma(x) dx = \frac{-\lambda y_0}{\pi} \int_{-\infty}^{\infty} \frac{dx}{(x^2 + y_0^2)} = \frac{-\lambda}{\pi} \left[ \arctan \frac{x}{y_0} \right]_{-\infty}^{\infty}$$

$$\lambda_z = \frac{-\lambda}{\pi} [\arctan(\infty) - \arctan(-\infty)] = -\lambda$$

Image with another line charge  
-λ below the mid-plane

