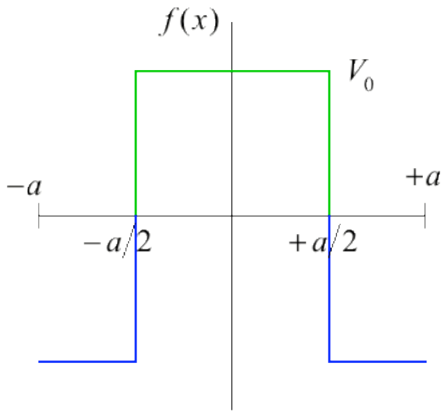
PHY481: Electromagnetism

Solving Laplace's equation via "Separation of Variables"

1) Cartesian coordinates
2) Spherical coordinates
3) Cylindrical coordinates

Coefficients in a Fourier Series



Find Fourier series expansion of $f(x) = V_0$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{a} + b_n \sin\frac{n\pi x}{a}\right]$$

$$a_m = \frac{1}{a}\int_{-a}^{a} f(x)\cos\frac{m\pi x}{a}$$

$$b_m = \frac{1}{a}\int_{-a}^{a} f(x)\sin\frac{m\pi x}{a} = 0$$

function f(x) is even so no sine terms

$$= \frac{1}{m\pi}\left[-V_0 \sin\frac{m\pi x}{a}\Big|_{-a}^{-a/2} + V_0 \sin\frac{m\pi x}{a}\Big|_{-a/2}^{+a/2} - V_0 \sin\frac{m\pi x}{a}\Big|_{+a/2}^{+a}\right]$$

Be careful to break integral into pieces !!

$$a_m = \begin{cases} \dfrac{4V_0}{m\pi}(-1)^n, & m = 2n+1 \text{ (odd)} \\ 0, & m = 2n \text{ (even)} \end{cases}$$

$$f(x) = \frac{4V_0}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\cos\left[(2n+1)\frac{\pi x}{a}\right]$$

Coordinate independence in separable solutions

Cartesian coordinate separable solutions $V(\mathbf{x}) = X(x)Y(y)Z(z)$

Translational independence x --> x′ $V(\mathbf{x}') = X(x')Y(y)Z(z)$

$$\nabla^2 V / V = \frac{1}{X(x')}\frac{d^2 X(x')}{dx^2} + \frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2 Z(z)}{dz^2} = 0$$

Values of the last two terms (Y and Z terms) have not changed.

Yet the sum is still ZERO. Each of the 3 terms must be a constant.

$$\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} = k_1^2 \qquad \frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2} = k_2^2 \qquad \frac{1}{Z(z)}\frac{d^2 Z(z)}{dz^2} = k_3^2$$

Solution to Laplace's equation must have $k_1^2 + k_2^2 + k_3^2 = 0$

Range of solutions

$$k_1^2 + k_2^2 + k_3^2 = 0$$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = k_1^2; \quad \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = k_2^2; \quad \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = k_3^2$$

If $k_1 = 0$ (simple field) function X satisfies

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = 0$$

solution: $X(x) = ax + b$

Constants a & b determined by the boundary conditions

If $k_1^2 > 0$, function $X(x)$ satisfies

$$\frac{d^2X(x)}{dx^2} = k_1^2 X(x)$$

solution: $X(x) = Ae^{k_1 x} + Be^{-k_1 x}$

etc., for Y & Z

If $k_1^2 < 0$, then $k \dashrightarrow ik'$ k' real > 0, and X satisfies

$$\frac{d^2X(x)}{dx^2} = -k_1'^2 X(x)$$

solution: $X(x) = Ae^{ik_1' x} + Be^{-ik_1' x}$

etc., for Y & Z

Character of solutions

$k_1^2 > 0$ $\quad X(x) = Ae^{k_1 x} + Be^{-k_1 x}$

Allowable values of A, B, and k_1 determined by symmetries and boundary conditions

If region unbounded +x $\quad A = 0, \quad X(x) = Be^{-k_1 x}$

If region unbounded -x $\quad B = 0, \quad X(x) = Ae^{k_1 x}$

If region bounded + & -

Even symmetry / Odd symmetry

$$X(x) = A'\cosh k_1 x + B'\sinh k_1 x$$

$$\cosh k_1 x = \left(\frac{e^{k_1 x} + e^{-k_1 x}}{2}\right); \quad \sinh k_1 x = \left(\frac{e^{k_1 x} - e^{-k_1 x}}{2}\right)$$

$k_1^2 < 0$

$$X(x) = Ae^{ik_1' x} + Be^{-ik_1' x}$$

Suggests using Fourier series for A, B, k_1

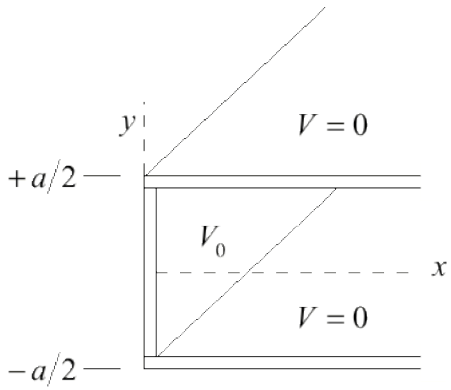
Even symmetry / Odd symmetry

$$X(x) = A'\cos k_1' x + B'\sin k_1' x$$

$$\cos k_1 x = \left(\frac{e^{ik_1 x} + e^{-ik_1 x}}{2}\right); \quad \sin k_1 x = \left(\frac{e^{ik_1 x} - e^{-ik_1 x}}{2i}\right)$$

Example: Long narrow channel

Large top and bottom grounded plates, potential V_0 on the end plate
Find potential everywhere between the plates.



$V(x,y)=X(x)Y(y)$ problems without z dependence all start the same way

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}+\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}=0$$

Laplace's equation

Each term must be a constant

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}=k_1^2 \qquad \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}=k_2^2$$

$$k_1^2+k_2^2=0$$

$$k_1^2=+k^2, \quad k_2^2=-k^2$$

Even y symmetry

<u>First determine Y</u>

$$\frac{d^2Y(y)}{dy^2}=-k^2Y(y)$$

$$Y(y)=A\cos ky+B\sin ky$$

$$Y(y)=A\cos ky \quad B=0$$

V = 0 on top and bottom plate

$$Y(a/2)=\cos(ka/2)=0$$

$$k=(2n+1)\frac{\pi}{a}; \quad n=0,\,1,\,2,\,\ldots$$

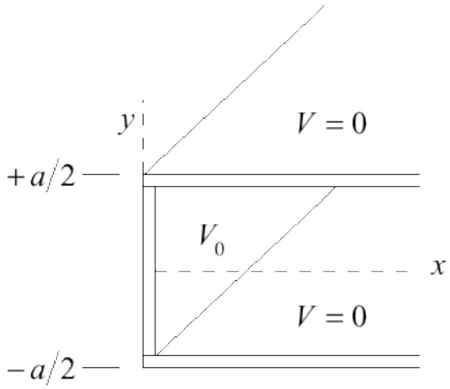
Any one of these values for k, satisfies the boundary condition at top and bottom, But an infinite series is needed to satisfy V(0,y) = V_0

Solution with coefficients

$$Y(y)=\sum_{n=0}^{\infty}A_n\cos\left[(2n+1)\frac{\pi y}{a}\right]$$

Example (cont'd)

Determine the X function



$$\frac{d^2X(x)}{dx^2}=+k^2X(x)$$

General solution $$X(x)=Ae^{kx}+Be^{-kx}$$

Finite at large +x $$X(x\to\infty)=0,\quad A=0$$ $$X(x)=Be^{-kx}$$

From Y solution $$k=(2n+1)\frac{\pi}{a};\quad n=0,\,1,\,2,\,\ldots$$ $$X(x)=Be^{-(2n+1)\pi x/a}$$

V = XY and determine the coefficients $$C_n=A_nB$$

$$V(x,y)=X(x)Y(y)=\sum_{n=0}^{\infty}C_n\cos\frac{(2n+1)\pi y}{a}e^{-(2n+1)\pi x/a}$$

On left boundary, Potential is V_0

$$V(0,y)=V_0=\sum_{n=0}^{\infty}C_n\cos\left[\frac{(2n+1)\pi y}{a}\right]$$

C_n by Fourier Integral

Be careful to break integral into pieces !!

$$f(y)=\begin{cases}V_0 & -a/2<y<a/2\\ -V_0 & \text{elsewhere}\end{cases}$$

$$C_n=\frac{1}{a}\left[\int_{-a}^{+a}f(y)\cos\frac{(2n+1)\pi y}{a}dy\right]=\frac{4V_0(-1)^n}{(2n+1)\pi}$$

Finally a solution $$V(x,y)=\frac{4V_0}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\cos\frac{(2n+1)\pi y}{a}e^{-(2n+1)\pi x/a}$$

Separation of Variables - spherical coordinates

Laplace's equation for potential in Spherical coordinates (no phi dep.)

$$\nabla^2 V(r,\theta) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)$$

Solutions are often of the "separable" type,

$$V(r,\theta) = R(r)\Theta(\theta)$$

or linear combinations of such solutions:

$$V(\mathbf{x}) = \sum_{\ell=0}^{\infty} V_\ell(\mathbf{x});$$

$$V_\ell(\mathbf{x}) = R_\ell(r)\Theta_\ell(\theta)$$

Inserting this form into Laplace's equation and multiplying by r^2/V

this depends only on r — this depends only on θ

$$\frac{r^2}{V}\nabla^2 V(r,\theta) = \underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)}_{+\ell(\ell+1)} + \underbrace{\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}_{-\ell(\ell+1)} = 0$$

Terms must be constants $+\ell(\ell+1)$ $-\ell(\ell+1)$ that sum to zero

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \ell(\ell+1)R = 0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\Theta = 0$$

Solving for functions $R(r)$ and $\Theta(\theta)$

$V(r,\theta) = R(r)\Theta(\theta)$ Separation of variables, solving Laplace's equation

Equation for R,

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \ell(\ell+1)R = 0$$

General solution

$$R(r) = Ar^\ell + \frac{B}{r^{\ell+1}}$$

non believers check this

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \ell(\ell+1)R$$

and equation for Θ

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\Theta = 0$$

Let $u = \cos\theta$, and $\Theta(\theta) = P(u)$

$$\frac{d\Theta}{d\theta} = \frac{du}{d\theta}\frac{dP}{du} = -\sin\theta\frac{dP}{du} = -\left(1-u^2\right)^{\frac{1}{2}}\frac{dP}{du}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = \frac{d}{du}\left(\left(1-u^2\right)\frac{dP}{du}\right)$$

Legendre's equation for $P(\cos\theta)$

$$\frac{d}{du}\left[\left(1-u^2\right)\frac{dP}{du}\right] + \ell(\ell+1)P = 0$$

Solutions are Legendre Polynomials: $P_\ell(\cos\theta)$

Finally,

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

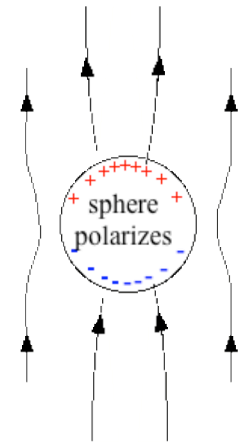
First 4 Legendre Polynomials

$$P_0(\cos\theta) = 1$$
$$P_1(\cos\theta) = \cos\theta$$
$$P_2(\cos\theta) = \left(3\cos^2\theta - 1\right)/2$$
$$P_3(\cos\theta) = \left(5\cos^3\theta - 3\cos\theta\right)/2$$

Familiar problem in spherical coordinates

Orthogonality of Legendre polynomials: $\int_{-1}^{1} P_m(\cos\theta) P_n(\cos\theta) d(\cos\theta) = \frac{2\delta_{\text{mn}}}{\text{2n+1}}$

Consider a grounded conducting sphere radius, *a*, in a constant external field, E_0 in z direction



Boundary condition V = 0 on sphere

$$V(a,\theta) = 0 = \sum_{\ell=0}^{\infty} \left(A_\ell a^\ell + B_\ell a^{-(\ell+1)} \right) P_\ell(\cos\theta)$$

Multiply by $P_n(\cos\theta)$ and integrate

$$\int_{-1}^{1} V(a,\theta) P_n(\cos\theta) d(\cos\theta) = 0 = \sum_{\ell=0}^{\infty} \left(A_\ell a^\ell + \frac{B_\ell}{a^{\ell+1}} \right) \int_{-1}^{1} P_\ell(\cos\theta) P_n(\cos\theta) d(\cos\theta)$$

$$0 = \left(A_n a^n + B_n a^{(n+1)} \right) 2/(2n+1) \qquad B_n = -A_n a^{2n+1}$$

Boundary condition

$$V(r \to \infty, \theta) = -E_0 z = -E_0 r \cos\theta$$

$$\cos\theta = P_1(\cos\theta) \qquad \Rightarrow \ell = 1 \qquad A_1 = -E_0$$

Same as solution found in Lecture 16

$$V(r,\theta) = \left(-E_0 r + \frac{E_0 a^3}{r^2} \right) \cos\theta$$

Cylindrical coordinates

Laplace's equation in cylindrical coordinates

$$\nabla^2 V(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Separation of variables

$$V(r,\phi) = R(r)\Phi(\phi)$$

<u>r dependence</u> <u>ϕ dependence</u>

$$\frac{r^2}{V}\nabla^2 V(r,\phi) = \frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

Constants sum to zero n^2 $-n^2$

Separate solutions

$$\Phi_n(\phi) = C_n \cos n\phi + D_n \sin n\phi$$

$$R_n(r) = A_n r^n + B_n r^{-n}$$

$$R_0(r) = A \ln r + B$$

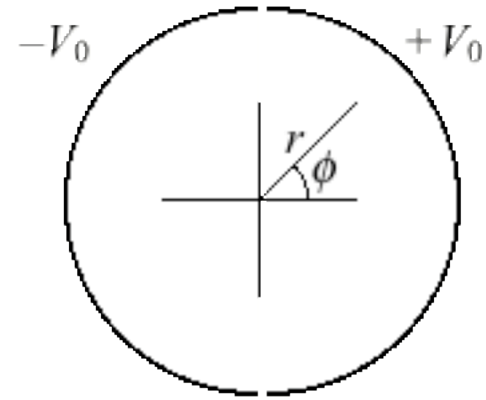
Solution with series to insure boundary conditions can be satisfied

$$V(r,\phi) = A\ln r + B + \sum_{n=1}^{\infty}\left(A_n r^n + B_n r^{-n}\right)\left(C_n \cos n\phi + D_n \sin n\phi\right)$$

Cylinder problem

Half cylinders, radius R. $V = +V_0$ on right half, and $V = -V_0$ on left half

Find potential inside and out.



$$V(r,\phi) = A\ln r + B + \sum_{n=1}^{\infty}\left(A_n r^n + B_n r^{-n}\right)\left(C_n \cos n\phi + D_n \sin n\phi\right)$$

Boundary conditions at r = 0 and r = ∞

$$V_{Int}(r,\phi) = \sum_{n=1}^{\infty} A_n r^n \cos n\phi$$

$$V_{Ext}(r,\phi) = \sum_{n=1}^{\infty} B_n r^{-n} \cos n\phi$$

$$V_{Int}(R,\phi) = V_{Ext}(R,\phi)$$

$$A_n R^n = B_n R^{-n} = c_n$$

$$A_n = \frac{c_n}{R^n}; \quad B_n = c_n R^n$$

Orthogonality

$$\int_0^{2\pi} \cos n\phi \cos m\phi = \pi\delta_{nm}$$

Use Orthogonality

Be careful to break Fourier integral into pieces !!

$$f(\phi) = \begin{cases} V_0 & 1^{\text{st}} \,\&\, 4^{\text{rd}} \text{ quadrants} \\ -V_0 & 2^{\text{nd}} \,\&\, 3^{\text{th}} \text{ quadrants} \end{cases}$$

$$V_{Int}(r,\phi) = \sum_{n=1}^{\infty} c_n \frac{r^n}{R^n} \cos n\phi$$

$$V_{Ext}(r,\phi) = \sum_{n=1}^{\infty} c_n \frac{R^n}{r^n} \cos n\phi$$

$$c_j = \frac{4V_0(-1)^j}{\pi(2j+1)} \sin\frac{(2j+1)\pi}{2}$$

Complete solution

$$V_{Int}(r,\phi) = \frac{4V_0}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\frac{r^{2j+1}}{R^{2j+1}}\cos(2j+1)\phi$$

$$V_{Ext}(r,\phi) = \frac{4V_0}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\frac{R^{2j+1}}{r^{2j+1}}\cos(2j+1)\phi$$