PHY481: Electromagnetism

Solving Laplace's equation via "Separation of Variables"

1) Cartesian coordinates ✓
2) Spherical coordinates
3) Cylindrical coordinates

Examples

Cartesian coordinates

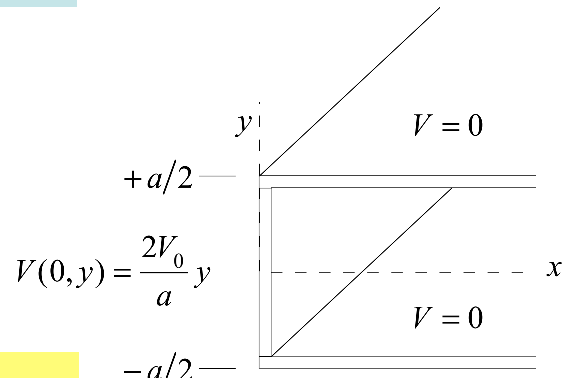
- Determine nature of solutions on the rectangle (unbounded z)
  - Grounded at $x = -a/2$ or $+a/2$ then $X_k(x) = A_k\cos(kx) + B_k\sin(kx)$
  - Grounded at $y = -a/2$ or $+a/2$ then $Y_k(y) = A_k\cos(ky) + B_k\sin(ky)$
  - Other solution is then $U_k(u) = C_k\cosh(ku) + D_k\sinh(ku)$ (same k)
- Use periodic (sine or cosine) solution and boundary conditions to determine legal values of $k = n\pi x/a$ (same for both solutions)
- Generate general solution $V(x,y) = \Sigma X_k(x)Y_k(y)$
- Use additional boundary conditions to determine coeficients
  - Periodic B.C.: $V_b(x_0,y) = \Sigma X_k(x_0)Y_k(y)$ or $V_b(x,y_0) = \Sigma X_k(x)Y_k(y_0)$
  - Apply Fourier Integral to both sides: $\frac{1}{a}\int_{-a}^{a} V_b(x,y_0)\cos m\pi x/a$ (or y version)

    Be sure to split Fourier integral (-a,-a/2,a/2,a)
  - Use orthogonality on right side: $\int_{-a}^{a}\cos(n\pi x/a)\cos(m\pi x/a) = \delta_{mn}$ (or y version)
  - Determine coef. A_k,B_k,C_k,D_k
- Substitute into $V(x,y) = \Sigma X_k(x)_kY(y)$

From Exam 2 last year

Boundary conditions $V(x,\pm a/2)=0,\ x\neq 0$ $V(0,y)=2V_0 y/a$ $V(x,y)\to 0,\ \text{as } x\to\infty$

Y Solution $Y(y)=A\cos(k\pi y/a)+B\sin(k\pi y/a)$



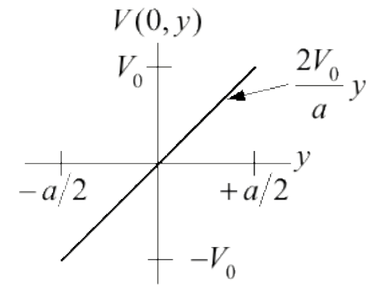
Apply boundary condition on Y solution

$V(0,y)=2V_0\,y/a$, an odd function $\Rightarrow\ A=0.$

$Y(\pm a/2)=B\sin(\pm k\pi/2)=0 \quad \Rightarrow\ k=2n$, an even integer.

Apply boundary condition on X solution (same k)

$X(x)=Ce^{2n\pi x/a}+De^{-2n\pi x/a};\quad V(x,y)\to 0,\ \text{as}\ x\to\infty,\quad \Rightarrow\ C=0$



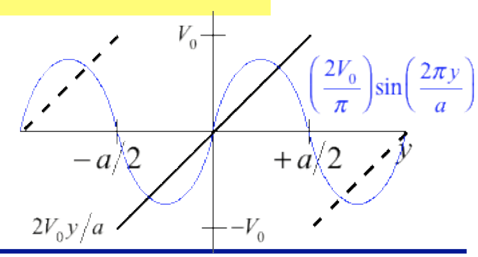
Most general solution consistent with boundary conditions

$$V(x,y)=\sum_{n=1}^{\infty}K_n\sin(2n\pi y/a)e^{-2n\pi x/a} \qquad V(0,y)=\sum_{n=1}^{\infty}K_n\sin(2n\pi y/a)$$

Determine coefficients

$$K_n=\frac{1}{a}\int_{-a}^{a}V(0,y)\sin(2n\pi y/a)\,dy=4\frac{2V_0}{a^2}\int_0^{a/2}y\sin(2n\pi y/a)\,dy=\frac{2V_0}{\pi}\left\{\frac{(-1)^{n+1}}{n}\right\}$$

Solution $$V(x,y)=\frac{2V_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(2n\pi y/a)e^{-2n\pi x/a}$$



Spherical coordinates: $V(r, \theta) = R(r)\Theta(\theta)$

Laplace's equation for potential in Spherical coordinates (no phi dep.)

$$\nabla^2 V(r,\theta) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right)$$

Sep. of variables

$$V(r,\theta) = R(r)\Theta(\theta)$$

Separate solutions

$$R(r) = Ar^{\ell} + \frac{B}{r^{\ell+1}}$$

$$\Theta(\theta) \to F(\cos\theta) = P_{\ell}(\cos\theta)$$

Legendre Polynomials

$$P_0(\cos\theta) = 1$$

$$P_1(\cos\theta) = \cos\theta$$

$$P_2(\cos\theta) = \left(3\cos^2\theta - 1\right)/2$$

$$P_3(\cos\theta) = \left(5\cos^3\theta - 3\cos\theta\right)/2$$

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_{\ell} r^{\ell} + \frac{B_{\ell}}{r^{\ell+1}}\right) P_{\ell}(\cos\theta)$$

Boundary condition on a sphere

$$V(a,\theta) = f(\theta)$$

$$\int_{-1}^{1} V(a,\theta) P_n(\cos\theta) d(\cos\theta)$$

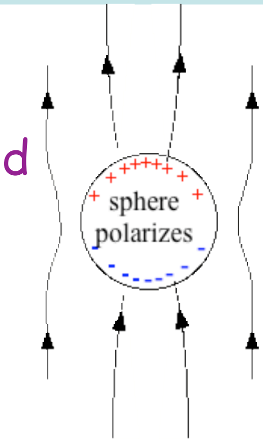
Orthogonality

$$\int_{-1}^{1} P_m(\cos\theta) P_n(\cos\theta) d(\cos\theta) = \frac{2\delta_{mn}}{2n+1}$$

Familiar problem in spherical coordinates

Orthogonality of Legendre polynomials: $\int_{-1}^{1} P_m(\cos\theta)P_n(\cos\theta)\,d(\cos\theta) = \frac{2\delta_{mn}}{2n+1}$

Consider a grounded conducting sphere radius, *a*, in a constant external field, E_0 in z direction



Boundary condition V = 0 on sphere

$$V(a,\theta) = 0 = \sum_{\ell=0}^{\infty}\left(A_\ell a^\ell + B_\ell a^{-(\ell+1)}\right)P_\ell(\cos\theta)$$

Multiply by $P_n(\cos\theta)$ and integrate

$$\int_{-1}^{1} V(a,\theta)P_n(\cos\theta)d(\cos\theta) = 0 = \sum_{\ell=0}^{\infty}\left(A_\ell a^\ell + \frac{B_\ell}{a^{\ell+1}}\right)\int_{-1}^{1} P_\ell(\cos\theta)P_n(\cos\theta)d(\cos\theta)$$

$$0 = \left(A_n a^n + B_n a^{-(n+1)}\right)2/(2n+1) \qquad B_n = -A_n a^{2n+1}$$

Boundary condition

$$V(r\to\infty,\theta) = -E_0 z = -E_0 r\cos\theta$$

$$\cos\theta = P_1(\cos\theta) \quad \boxed{\Rightarrow \ell = 1} \quad A_1 = -E_0$$

uniform field dipole dipole moment

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta) \longrightarrow = -E_0 r\cos\theta + \frac{E_0 a^3\cos\theta}{r^2}$$

$$E_0 a^3 = \frac{p}{4\pi\varepsilon_0}$$

Cylindrical coordinates

Laplace's equation in cylindrical coordinates

$$\nabla^2 V(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Separation of variables

$$V(r,\phi) = R(r)\Phi(\phi)$$

r dependence ϕ dependence

$$\frac{r^2}{V}\nabla^2 V(r,\phi) = \frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

Constants sum to zero n^2 $-n^2$

Separate solutions

$$\Phi_n(\phi) = C_n \cos n\phi + D_n \sin n\phi$$

$$R_n(r) = A_n r^n + B_n r^{-n}$$

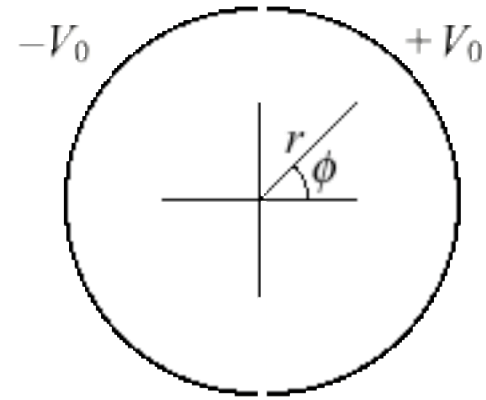
$$R_0(r) = A \ln r + B$$

Solution with series to insure boundary conditions can be satisfied

$$V(r,\phi) = A\ln r + B + \sum_{n=1}^{\infty}\left(A_n r^n + B_n r^{-n}\right)\left(C_n \cos n\phi + D_n \sin n\phi\right)$$

Cylinder problem

Half cylinders, radius R. $V = +V_0$ on right half, and $V = -V_0$ on left half
Find potential inside and out.



$$V(r,\phi) = A\ln r + B + \sum_{n=1}^{\infty}\left(A_n r^n + B_n r^{-n}\right)\left(C_n \cos n\phi + D_n \sin n\phi\right)$$

Boundary conditions at r = 0 and r = ∞

$$V_{Int}(r,\phi) = \sum_{n=1}^{\infty} A_n r^n \cos n\phi$$

$$V_{Ext}(r,\phi) = \sum_{n=1}^{\infty} B_n r^{-n} \cos n\phi$$

$$V_{Int}(R,\phi) = V_{Ext}(R,\phi)$$

$$A_n R^n = B_n R^{-n} = c_n$$

$$A_n = \frac{c_n}{R^n}; \quad B_n = c_n R^n$$

Orthogonality

$$\int_0^{2\pi} \cos n\phi \cos m\phi = \pi\delta_{nm}$$

Use Orthogonality

Be careful to break Fourier integral into pieces !!

$$f(\phi) = \begin{cases} V_0 & 1^{\text{st}} \,\&\, 4^{\text{rd}} \text{ quadrants} \\ -V_0 & 2^{\text{nd}} \,\&\, 3^{\text{th}} \text{ quadrants} \end{cases}$$

$$V_{Int}(r,\phi) = \sum_{n=1}^{\infty} c_n \frac{r^n}{R^n} \cos n\phi$$

$$V_{Ext}(r,\phi) = \sum_{n=1}^{\infty} c_n \frac{R^n}{r^n} \cos n\phi$$

$$c_j = \frac{4V_0(-1)^j}{\pi(2j+1)} \sin\frac{(2j+1)\pi}{2}$$

Complete solution

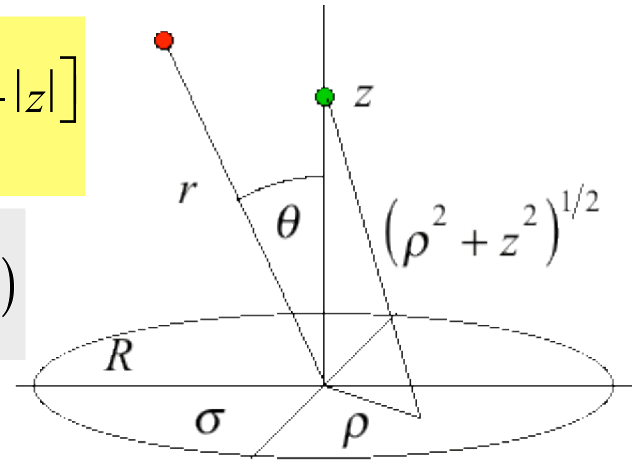
$$V_{Int}(r,\phi) = \frac{4V_0}{\pi}\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \frac{r^{2j+1}}{R^{2j+1}} \cos(2j+1)\phi$$

$$V_{Ext}(r,\phi) = \frac{4V_0}{\pi}\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \frac{R^{2j+1}}{r^{2j+1}} \cos(2j+1)\phi$$

Spherical problem

Disk, charge density σ, radius *R*. Determine potential on z-axis.

$$V(r,0)=V(z,0)=\frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi}d\phi\int_0^R\frac{\rho\,d\rho}{\left(z^2+\rho^2\right)^{1/2}}=\frac{\sigma}{2\varepsilon_0}\left[(z^2+R^2)^{1/2}-|z|\right]$$



Use an expansion of it to find potential everywhere.

$$V(r,\theta)=\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

$$(z^2+R^2)^{1/2}=z\left(1+R^2/z^2\right)^{1/2}=z\left(1+R^2/2z^2-R^4/8z^4+\ldots\right)$$

$$V(z,0)=\frac{\sigma}{2\varepsilon_0}\left[(z^2+R^2)^{1/2}-|z|\right]=\frac{\sigma R^2}{2\varepsilon_0}\left[\frac{1}{2z}-\frac{R^2}{8z^3}+\ldots\right]$$

$$f(\varepsilon)=1+f'(0)\varepsilon+f''(0)\varepsilon^2/2$$

$$f=(1+\varepsilon)^{1/2}\quad;f(0)=1$$
$$f'=(1+\varepsilon)^{-1/2}/2\quad;f'(0)=1/2$$
$$f''=-(1+\varepsilon)^{-1/2}/4\,;f''(0)=-1/4$$

$$(1+\varepsilon)^{1/2}=1+\varepsilon/2-\varepsilon^2/8$$

$$(1+x^2)^{1/2}=1+x^2/2-x^4/8+\ldots$$

Match coefficients

All $A_\ell=0$

$$V(r,\theta)=\frac{\sigma R^2}{2\varepsilon_0}\left\{\left(\frac{B_0}{r^1}\right)P_0(\cos\theta)+\left(\frac{B_2}{r^3}\right)P_2(\cos\theta)\right\}$$

$$B_0=\frac{1}{2};B_2=-\frac{R^2}{8}$$

$$V(r,\theta)=\frac{\sigma\pi R^2}{4\pi\varepsilon_0}\left\{\left(\frac{1}{r}\right)-\left(\frac{R^2}{4r^3}\right)\left(\frac{3\cos\theta-1}{2}\right)+\ldots\right\}$$