PHY481: Electromagnetism

HW5

5.3

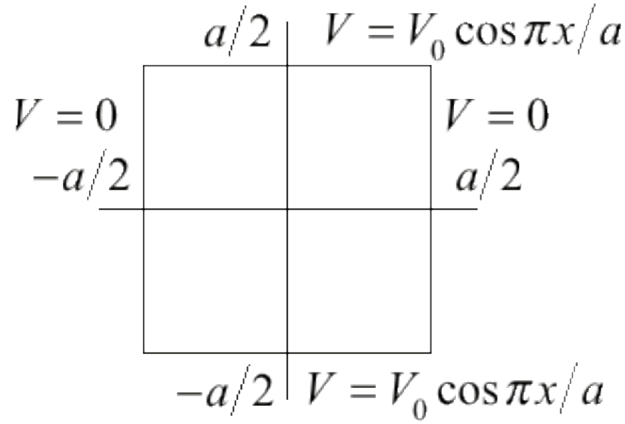
Boundary condition $V = V_0 \cos\left(\frac{\pi x}{a}\right)$ at $y = \pm a/2$

General solution (for even boundary conditions)

$$V(x,y) = C_n \cos\left[(2n+1)\frac{\pi x}{a}\right]\cosh\left[(2n+1)\frac{\pi y}{a}\right]$$

$$V(x,\pm a/2) = V_0 \cos\left(\frac{\pi x}{a}\right)$$
$$= C_n \cos\left[(2n+1)\frac{\pi x}{a}\right]\cosh\left[(2n+1)\frac{\pi y}{a}\right]$$

<u>Square pipe cross section</u>



Only non-zero term

$n = 0,\ C_n = C_0$

Apply boundary condition to determine coefficient

$$V(x,\pm a/2) = C_0 \cos\left[\frac{\pi x}{a}\right]\cosh\left[\frac{\pi}{2}\right] = V_0 \cos\left[\frac{\pi x}{a}\right]$$

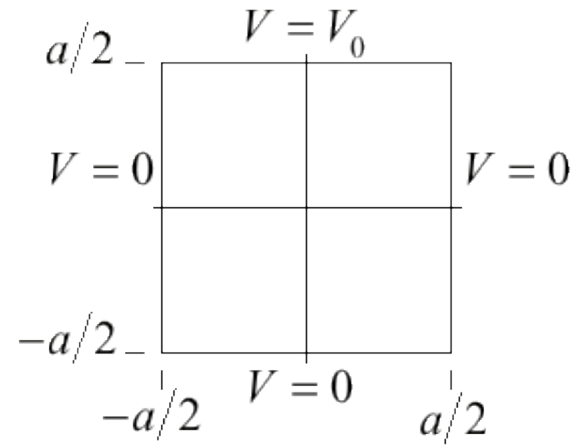
$$C_0 = \frac{V_0}{\cosh\left[\frac{\pi}{2}\right]}$$

Potential inside pipe

$$V(x,y) = \frac{V_0}{\cosh(\pi/2)}\cos(\pi x/a)\cosh(\pi y/a)$$

5.7

$$X(x) = B_n \cos[(2n+1)]\pi x/a$$ boundary conditions: $X(\pm a/2) = 0$



$$Y(y) = C_n \sinh[(2n+1)\pi y/a] + D_n \cosh[(2n+1)\pi y/a]$$
$$= C_n \sinh[(2n+1)\pi y/a + \phi_n]$$ instead use phase: ϕ_n

$$V(x,y) = \sum_{n=0}^{\infty} g_n \sinh[(2n+1)\pi\, y/a + \phi_n]\cos[(2n+1)\pi x/a]$$

$$V(x,-a/2) = 0 = \sum_{n=0}^{\infty} g_n \sinh[-(2n+1)\pi/2 + \phi_n]\cos[(2n+1)\pi x/a]$$

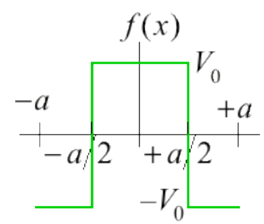
bc: $Y(-a/2) = 0$ sets phase

$$\phi_n = (2n+1)\pi/2$$

$$V(x,+a/2) = V_0 = \sum_{n=0}^{\infty} g_n \sinh[(2n+1)\pi]\cos[(2n+1)\pi x/a]$$

bc: $Y(+a/2) = V_0$ sets coef. g_n

$$\frac{1}{a}\int_{-a}^{+a} f(x)\cos[(2n+1)\pi x/a] = g_n \sinh[(2n+1)\pi]$$



$$g_n = \frac{4V_0(-1)^n}{(2n+1)\pi}\frac{1}{\sinh[(2n+1)\pi]}$$

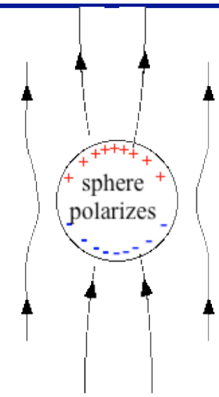
Be careful to break integral into pieces !!

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\frac{\sinh[(2n+1)\pi(y/a+1/2)]}{\sinh[(2n+1)\pi]}\cos[(2n+1)\pi x/a]$$

5.11

See lecture 18 for derivation of potential for on a conducting sphere in an external field

$$V(r,\theta) = +\frac{E_0 a^3 \cos\theta}{r^2} - E_0 r\cos\theta$$



$$\mathbf{r} = a\cos\theta\,\hat{\mathbf{k}} + a\sin\theta\,\hat{\boldsymbol{\rho}}$$

Dipole potential

$$V(r,\theta) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

Compare $1/r^2$ potentials

$$p = 4\pi\varepsilon_0 E_0 a^3$$

Dipole moment analogous to $p = qd$

$$\mathbf{p} = \int \sigma(\theta)\mathbf{r}\,dS$$

$$= 2\pi a^3 \int \left(3\varepsilon_0 E_0 \cos\theta\right)\cos\theta\sin\theta\,d\theta\,\hat{\mathbf{k}}$$

Polarizability

$$p = \alpha E_0$$

$$\alpha = 4\pi\varepsilon_0 a^3$$

$$p_z = 6\pi a^3 \varepsilon_0 E_0 \int_0^{\pi} \cos^2\theta\sin\theta\,d\theta$$

$$= 6\pi a^3 \varepsilon_0 E_0 \int_1^{-1} -u^2\,du$$

$$\sigma(\theta) = \varepsilon_0 E_r(a,\theta) = -\varepsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=a}$$

$$= 2\varepsilon_0 \frac{E_0 a^3 \cos\theta}{a^3} + \varepsilon_0 E_0 \cos\theta = 3\varepsilon_0 E_0 \cos\theta$$

Charge density

$$\sigma(\theta) = 3\varepsilon_0 E_0 \cos\theta$$

Dipole moment

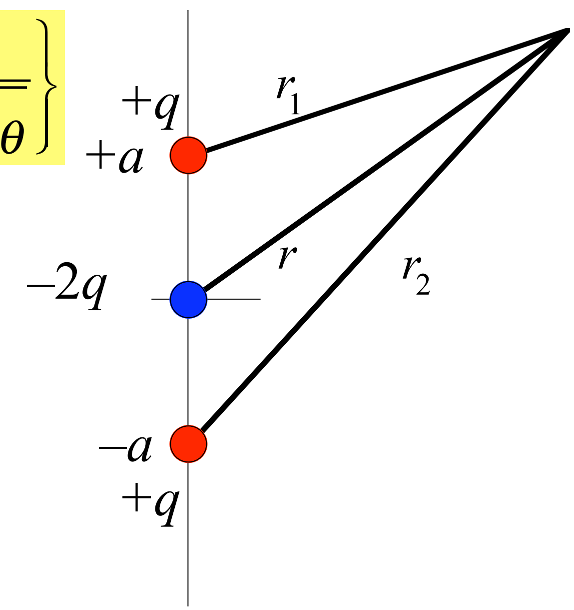
$$p_z = 4\pi a^3 \varepsilon_0 E_0$$

5.13

Linear quadrupole exact potential

$$V(r,\theta)=\frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}}+\frac{-2}{r}+\frac{1}{\sqrt{r^2+a^2+2ar\cos\theta}}\right\}$$

Linear quadrupole



Approximation

Expansions

$$\frac{1}{\sqrt{1+\varepsilon}}=1-\frac{1}{2}\varepsilon+\frac{3}{8}\varepsilon^2+\ldots$$

$$\frac{1}{\sqrt{r^2+a^2\pm 2ar\cos\theta}}=\frac{1}{r}\left\{1\pm\frac{a\cos\theta}{r}+\frac{3a^2\cos^2\theta-a^2}{2r^2}+\ldots\right\}$$

Monopole and dipole terms cancel

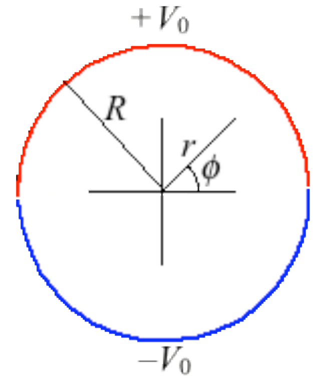
$$V(r,\theta)=\frac{qa^2}{4\pi\varepsilon_0}\left\{\frac{3\cos^2\theta-1}{2r^3}+\ldots\right\}=\frac{2qa^2}{4\pi\varepsilon_0}\frac{P_2(\cos\theta)}{r^3}+\ldots$$

5.15 (see lecture 19)

Half cylinders, radius R. $V = +V_0$ on upper half, and $V = -V_0$ on lower half

Find potential inside and out. (odd function of ϕ -> no cosine terms)

$$V(r,\phi) = A\ln r + B + \sum_{n=1}^{\infty}\left(A_n r^n + B_n r^{-n}\right)\left(C_n \cos n\phi + D_n \sin n\phi\right)$$



B.C. at r = 0 and r = ∞

$$V_{Int}(r,\phi) = \sum_{n=1}^{\infty} A_n r^n \sin n\phi$$

$$V_{Ext}(r,\phi) = \sum_{n=1}^{\infty} B_n r^{-n} \sin n\phi$$

B.C. at r = R

$$V_{Int}(R,\phi) = V_{Ext}(R,\phi)$$

$$A_n R^n = B_n R^{-n} = c_n \to A_n = \frac{c_n}{R^n}; \quad B_n = c_n R^n$$

Fourier Integral at r = R

Be careful to break Fourier integral into pieces !!

Use Orthogonality

$$\int_0^{2\pi} \cos n\phi \cos m\phi = \pi\delta_{nm}$$

$$\int_0^{2\pi} V(r,\phi)\sin m\phi d\phi = \sum_{n=1}^{\infty} c_n \int_0^{2\pi} \sin n\phi \sin m\phi d\phi$$

$$-V_0 \cos m\phi\Big|_0^{\pi} + V_0 \cos m\phi\Big|_{\pi}^{2\pi} = c_n \pi\delta_{mn}$$

$$c_m = \frac{4V_0}{\pi m} \quad m = 1,3,5...$$

Complete solution

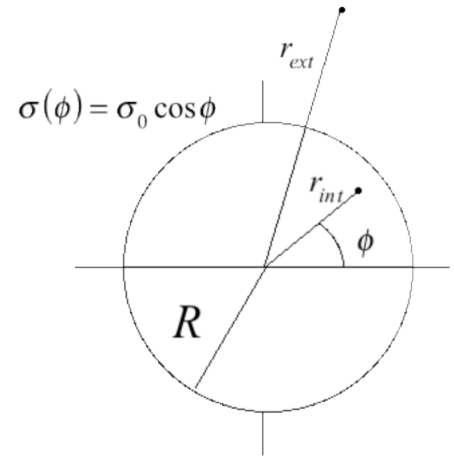
$$V_{Int}(r,\phi) = \frac{4V_0}{\pi}\sum_{m \text{ odd}}^{\infty} \frac{1}{m}\frac{r^m}{R^m}\sin m\phi$$

$$V_{Ext}(r,\phi) = \frac{4V_0}{\pi}\sum_{m \text{ odd}}^{\infty} \frac{1}{m}\frac{R^m}{r^m}\sin m\phi$$

5.16

General solution in cylindrical coordinates

$$V(r,\phi) = A\ln r + B + \sum_{n=1}^{\infty}\left(A_n r^n + B_n r^{-n}\right)\left(C_n \cos n\phi + D_n \sin n\phi\right)$$



$$E_r(R_{ext}) - E_r(R_{\text{int}}) = \frac{\sigma}{\varepsilon_0}$$

Boundary condition crossing the boundary

$$V(r_{\text{int}},\phi) = Ar\cos\phi \qquad E_n(r_{\text{int}},\phi) = -A\cos\phi$$

$$V(r_{ext},\phi) = \frac{B}{r}\cos\phi \qquad E_n(r_{ext},\phi) = +\frac{B\cos\phi}{r_{ext}^2}$$

Matching terms

Apply boundary conditions

$$\frac{B\cos\phi}{r_{ext}^2} + A\cos\phi = \frac{\sigma_0\cos\phi}{\varepsilon_0}$$

$$\frac{B}{R^2} + A = \frac{\sigma_0}{\varepsilon_0}$$

$$V_{\text{int}}(R,\phi) = V_{ext}(R,\phi)$$

$$AR\cos\phi = \frac{B}{R}\cos\phi$$

$$AR = \frac{B}{R}$$

$$A = \frac{\sigma_0}{2\varepsilon_0} \qquad B = \frac{\sigma_0 R^2}{2\varepsilon_0}$$

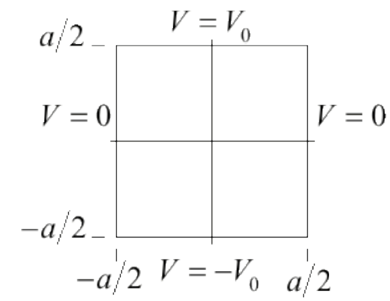
Solution inside and out

$$V(r_{\text{int}},\phi) = \frac{\sigma_0}{2\varepsilon_0} r\cos\phi; \quad V(r_{ext},\phi) = \frac{\sigma_0 R^2}{2\varepsilon_0 r}\cos\phi$$

5.28 (see problem 5.7)

$X(x)=B_n\cos[(2n+1)]\pi x/a$ boundary conditions: $X(\pm a/2)=0$

$Y(y)=C_n\sinh[(2n+1)\pi y/a]$ odd B. C. : $Y(\pm a/2)=\pm V_0$



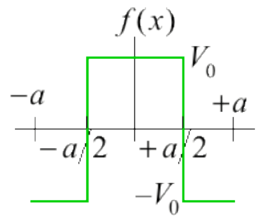
$$V(x,y)=\sum_{n=0}^{\infty}g_n\sinh[(2n+1)\pi y/a]\cos[(2n+1)\pi x/a]$$ $g_n=B_nC_n$

$$V(x,+a/2)=V_0=\sum_{n=0}^{\infty}g_n\sinh[(2n+1)\pi/2]\cos[(2n+1)\pi x/a]$$

B.C.: $Y(+a/2)=V_0$ sets coef. g_n

$$\frac{1}{a}\int_{-a}^{+a}f(x)\cos[(2n+1)\pi x/a]=g_n\sinh[(2n+1)\pi/2]$$

Be careful to break Fourier integral into pieces !!



$$\frac{V_0}{(2n+1)\pi}\left\{\begin{array}{l}-\sin[(2n+1)\pi x/a]_{-a}^{-a/2}+\sin[(2n+1)\pi x/a]_{-a/2}^{a/2}\\ -\sin[(2n+1)\pi x/a]_{a/2}^{a}\end{array}\right\}=g_n\sinh[(2n+1)\pi/2]$$

$$g_n=\frac{4V_0(-1)^n}{(2n+1)\pi\sinh[(2n+1)\pi/2]}$$

$$V(x,y)=\frac{4V_0}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\frac{\sinh[(2n+1)\pi y/a]}{\sinh[(2n+1)\pi/2]}\cos[(2n+1)\pi x/a]$$

5.32

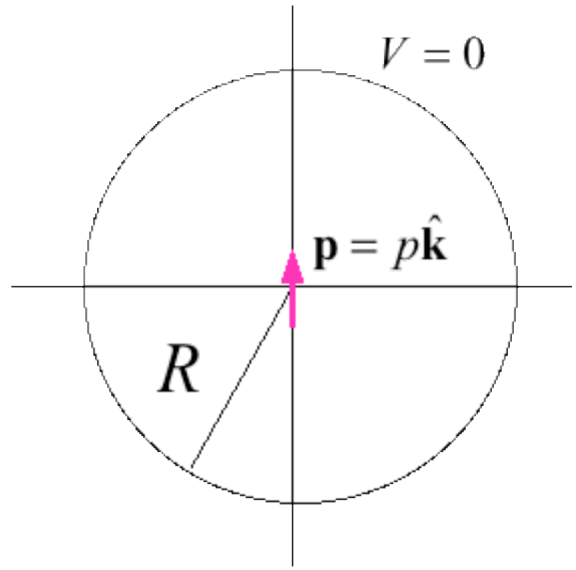
Dipole potential

$$V = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

General solution in spherical coordinates

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

Dipole in grounded conducting sphere



$$V(r,\theta) = \left(Ar + \frac{B}{r^2}\right)P_1(\cos\theta)$$

Only one term $\ell = 1$

Apply boundary conditions

r -> 0 $$V(r,\theta) \to \frac{B\cos\theta}{r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$ $$B = \frac{p}{4\pi\varepsilon_0}$$

r = R $$V(R,\theta) = \left(AR + \frac{B}{R^2}\right) = 0$$

$$A = \frac{-B}{R^3} = \frac{-p}{4\pi\varepsilon_0 R^3}$$

Full Solution: $$V(r,\theta) = \frac{p}{4\pi\varepsilon_0}\left(\frac{-r}{R^3} + \frac{1}{r^2}\right)\cos\theta$$

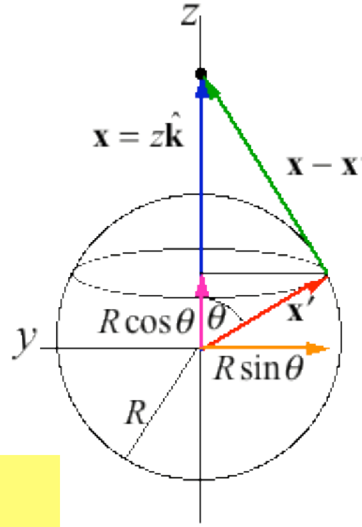
1

Surface charge density $\sigma(\theta) = \sigma_0 \cos\theta$ find Potential inside and out

$$V(z) = \int \frac{\rho(\mathbf{x}')\,d^3x'}{|\mathbf{x}-\mathbf{x}'|} = \frac{\sigma_0 R^2}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi \int_0^{\pi} \frac{\cos\theta \sin\theta\, d\theta}{\sqrt{R^2+z^2-2Rz\cos\theta}} = \frac{\sigma_0 R^2}{2\varepsilon_0}\int_{-1}^{1}\frac{x\,dx}{\sqrt{R^2+z^2-2Rzx}}$$

$$\rho(\mathbf{x}')\,d^3x' = \sigma(\theta)\,d^2x' = (\sigma_0\cos\theta)\,R^2\sin\theta\, d\theta\, d\phi$$

$$|\mathbf{x}-\mathbf{x}'| = \sqrt{(z-R\cos\theta)^2 + R^2\sin^2\theta} = \sqrt{z^2+R^2-2Rz\cos\theta}$$



$$\int_{-1}^{1}\frac{x\,dx}{\sqrt{R^2+z^2-2Rzx}} = \frac{1}{3R^2z^2}\left[-\sqrt{R^2-2Rxz+z^2}\,(R^2+Rxz+z^2)\right]_{-1}^{1} = \frac{1}{3R^2z^2}\left[-\sqrt{(R-z)^2}\left[R^2+Rz+z^2\right]+\sqrt{(R+z)^2}\left[R^2-Rz+z^2\right]\right]$$

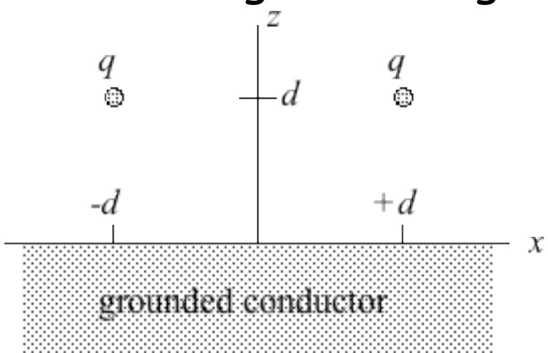
for $z < R$ $\quad = \left\{-(R-z)\left[R^2+Rz+z^2\right]+(R+z)\left[R^2-Rz+z^2\right]\right\}/3R^2z^2 = 2z/3R^2$

for $z > R$ $\quad = \left\{-(z-R)\left[R^2+Rz+z^2\right]+(R+z)\left[R^2-Rz+z^2\right]\right\}/3R^2z^2 = 2R/z^2$

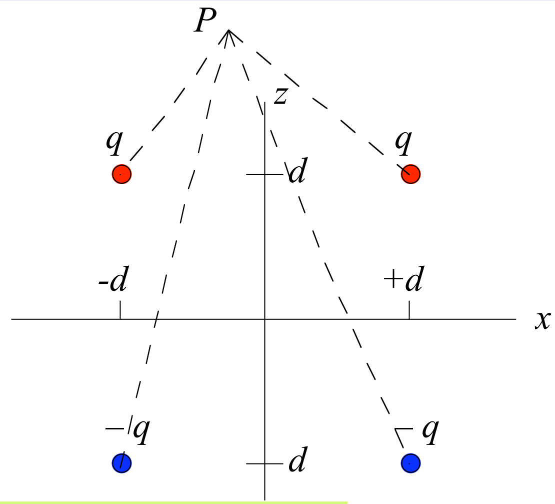
$$V(z) = \frac{\sigma_0}{3\varepsilon_0}z \quad \text{for } z < R \qquad\qquad V(z) = \frac{\sigma_0 R^3}{3\varepsilon_0 z^2} \quad \text{for } z > R$$

2

Two charges above grounded plane



Image

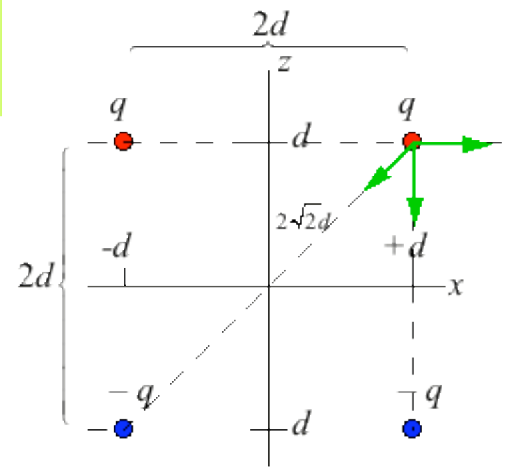


Potential above the plane

$$V(x,0,z) = \frac{q}{4\pi\varepsilon_0}\left\{\begin{array}{l} \dfrac{1}{\sqrt{(x-d)^2+(z-d)^2}} + \dfrac{1}{\sqrt{(x+d)^2+(z-d)^2}} \\ \quad - \dfrac{1}{\sqrt{(x-d)^2+(z+d)^2}} - \dfrac{1}{\sqrt{(x+d)^2+(z+d)^2}} \end{array}\right\}$$

$$\mathbf{F} = \frac{q^2}{4\pi\varepsilon_0}\left\{\frac{\hat{\mathbf{i}}}{4d^2} - \frac{\hat{\mathbf{k}}}{4d^2} - \frac{\hat{\mathbf{i}}+\hat{\mathbf{k}}}{8d^2\sqrt{2}}\right\}$$

$$= \frac{q^2}{16\pi\varepsilon_0 d^2}\left\{\left(1-\frac{\sqrt{2}}{4}\right)\hat{\mathbf{i}} - \left(1+\frac{\sqrt{2}}{4}\right)\hat{\mathbf{k}}\right\}$$

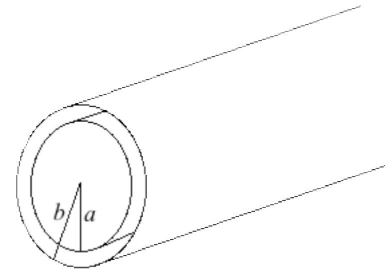


Cylindrical capacitor $d = b - a << a$

a) Means spacing very small

b) Means cylinder areas are nearly the same

c) Field between the cylinders must be ~ the field between a parallel plates with the same area.



Initial state

$$C = \frac{\varepsilon_0 A}{d} = \frac{2\pi\varepsilon_0 aL}{b-a}$$ $$Q = \frac{2\pi\varepsilon_0 aL}{b-a}V$$ $$U = \frac{1}{2}\frac{Q^2}{C} = \frac{Q^2(b-a)}{4\pi\varepsilon_0 aL}$$

Pull inner cylinder out changing the overlap $dL = L' - L$ (negative)

$$\frac{dU}{U} = -\frac{dL}{L}$$ $$W = dU = -U\frac{dL}{L}$$ $$\frac{dU}{dL} = -\frac{U}{L}$$ $$F = -\frac{dU}{dL} = \frac{U}{L} = \frac{Q^2(b-a)}{4\pi\varepsilon_0 aL^2}$$

Without initial assumptions

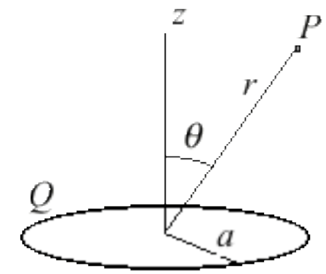
$$V(r) = \frac{V_0 \ln(b/r)}{\ln(b/a)}$$ $$E(r) = \frac{V_0}{\ln(b/a)r}$$ $$E(a) = \frac{V_0}{\ln(b/a)a} = \frac{\sigma}{\varepsilon_0} = \frac{Q}{2\pi\varepsilon_0 aL}$$

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon_0 L}{\ln(b/a)} \approx \frac{2\pi\varepsilon_0 aL}{b-a}$$ because $\ln(b/a) = \ln[(a+d)/a] = \ln[1 + a/d] \approx (b-a)/a$

4

Potential on z axis

$$V(z)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|}=\frac{Q}{4\pi\varepsilon_0}\left(z^2+a^2\right)^{-1/2}=\frac{Q}{4\pi\varepsilon_0 z}\left(1+a^2/z^2\right)^{-1/2}$$



Expansion in powers of a/z let $\delta=\varepsilon^2=a^2/z^2$

$$f(\delta)=(1+\delta)^{-1/2};\ f'(\delta)=-\tfrac{1}{2}(1+\delta)^{-3/2};\ f''(\delta)=+\tfrac{3}{4}(1+\delta)^{-5/2}$$

$$V(z)=\frac{Q}{4\pi\varepsilon_0 z}\left(f(0)+f'(0)\frac{a^2}{z^2}+f''(0)\frac{a^4}{2z^4}+\ldots\right)$$

$$=\frac{Q}{4\pi\varepsilon_0 z}\left(1-\frac{a^2}{2z^2}+\frac{3a^4}{8z^4}+\ldots\right)=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{z}-\frac{a^2}{2z^3}+\frac{3a^4}{8z^5}+\ldots\right)$$

Potential in spherical coordinates with boundary condition at large r

$$V(r\to\infty,\theta)=0\quad\Rightarrow A_\ell=0\qquad V(r,\theta)=\sum_{\ell=0}^{\infty}B_\ell r^{-(\ell+1)}P_\ell(\cos\theta)$$

Matching coefficients for solutions on the z-axis

$$V(z,0)=\sum_{\ell=0}^{\infty}B_\ell z^{-(\ell+1)}=\frac{B_0}{z}+\frac{B_2}{z^3}+\frac{B_4}{z^5}+\ldots=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{z}-\frac{a^2}{2z^3}+\frac{3a^4}{8z^5}+\ldots\right)$$

4(cont'd

Matching coefficients for solutions on the z-axis

$$V(z,0)=\sum_{\ell=0}^{\infty}B_\ell z^{-(\ell+1)}=\frac{B_0}{z}+\frac{B_2}{z^3}+\frac{B_4}{z^5}+\ldots=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{z}-\frac{a^2}{2z^3}+\frac{3a^4}{8z^5}+\ldots\right)$$

No odd ℓ solutions

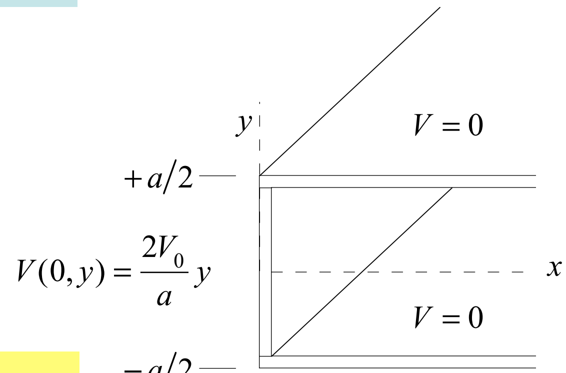
$$B_\ell=0 \text{ for odd } \ell,\ B_0=\frac{Q}{4\pi\varepsilon_0};\quad B_2=\frac{Q}{4\pi\varepsilon_0}\frac{a^2}{2};\quad B_4=\frac{Q}{4\pi\varepsilon_0}\frac{3a^4}{8}$$

Insert coefficients into general solution

$$V(r,\theta)=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r}-\frac{a^2}{2r^3}P_2(\cos\theta)+\frac{3a^4}{8r^5}P_4(\cos\theta)+\ldots\right)$$

Boundary conditions $V(x,\pm a/2)=0,\ x\neq 0$ $V(0,y)=2V_0 y/a$ $V(x,y)\to 0,\ \text{as } x\to\infty$

Y Solution $Y(y)=A\cos(k\pi\, y/a)+B\sin(k\pi\, y/a)$



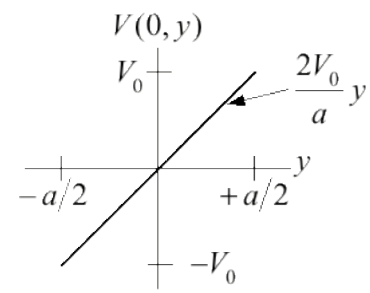
Apply boundary condition on Y solution

$V(0,y)=2V_0\, y/a$, an odd function $\Rightarrow$ $A=0$.

$Y(\pm a/2)=B\sin(\pm k\,\pi/2)=0 \quad \Rightarrow$ $k=2n$, an even integer.

Apply boundary condition on X solution (same *k*)

$X(x)=Ce^{2n\pi x/a}+De^{-2n\pi x/a};\quad V(x,y)\to 0,\ \text{as}\ x\to\infty,\quad \Rightarrow\ C=0$



Most general solution consistent with boundary conditions

$$V(x,y)=\sum_{n=1}^{\infty}K_n\sin(2n\pi\, y/a)e^{-2n\pi x/a} \qquad V(0,y)=\sum_{n=1}^{\infty}K_n\sin(2n\pi\, y/a)$$

Determine coefficients

$$K_n=\frac{1}{a}\int_{-a}^{a}V(0,y)\sin(2n\pi\, y/a)\,dy=4\frac{2V_0}{a^2}\int_0^{a/2}y\sin(2n\pi\, y/a)\,dy=\frac{2V_0}{\pi}\left\{\frac{(-1)^{n+1}}{n}\right\}$$

Solution $$V(x,y)=\frac{2V_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(2n\pi y/a)e^{-2n\pi x/a}$$