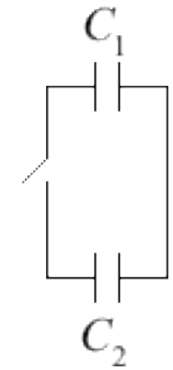
PHY481: Electromagnetism

Constant currents

Exam problem 2

Capacitors C_1, and C_2 carry charges Q_1 and Q_2 respectively.
The switch is closed causing the charge to redistribute



a) What are the new charges Q_1' and Q_2' ?

Charge is conserved: $Q_1' + Q_2' = Q_1 + Q_2$

Final potential is the same: $\dfrac{Q_1'}{C_1} = \dfrac{Q_2'}{C_2} = V'$ $\quad Q_2' = \dfrac{C_2}{C_1} Q_1'$

$$Q_1' = \underline{(Q_1 + Q_2) \Big/ \left(1 + \frac{C_2}{C_1}\right)}; \quad Q_2' = \underline{(Q_1 + Q_2) \Big/ \left(1 + \frac{C_1}{C_2}\right)}$$

a) What happens to the stored energy? $Q_1' = Q_1 - \Delta Q; \quad Q_2' = Q_2 + \Delta Q$

$$U_1' = \frac{1}{2}\frac{Q_1'^2}{C_1} = \frac{1}{2}\frac{(Q_1 - \Delta Q)^2}{C_1}; \quad U_2' = \frac{1}{2}\frac{Q_2'^2}{C_2} = \frac{1}{2}\frac{(Q_2 + \Delta Q)^2}{C_2}$$

$$U_1' + U_2' = U_1 + U_2 + \Delta Q\left(\frac{Q_2}{C_2} - \frac{Q_1}{C_1}\right) + \frac{(\Delta Q)^2}{2}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

$$\Delta U = -\frac{1}{2}\frac{(\Delta Q)^2}{C}$$

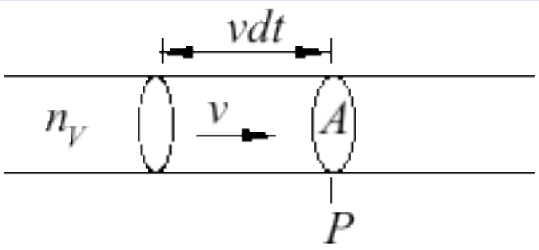
Charge carriers

$n_V =$ no. of charge carriers per unit volume

Charge passing P per unit time

$$dQ = qn_V Avdt$$

$$I = \frac{dQ}{dt} = qn_V Av$$



$n_L =$ no. of charge carriers per unit length

$$I = qn_L v \qquad v = v_{drift}$$

Examples

Cu, atomic mass 63, mass density 9 g/cm^3.

Assume 1 charge carrier (assume an electron) per Cu atom.

How many charge carriers in 1 cm^3 of Cu ?

$$n_V = (6\times10^{23} / 63\text{ g})(9\text{ g/cm}^3) = 9\times10^{22} / \text{cm}^3$$

right answer!
(but rather lucky)

How big is a Coulomb?

$$q = 1.6\times10^{-19}\text{ C}$$

Wire 1 mm^2 area. How long a wire has 1 Coulomb of charge carriers?

$$Q = (1.6\times10^{-19}\text{C})(9\times10^{22}\text{ cm}^{-3})(1\times10^{-2}\text{ cm}^2)L \qquad \Rightarrow \quad L = 0.07\text{ mm}$$

The wire carries a current of 1 A (1 C/s). What is the speed of the carriers?

$$v_{drift} = 0.07\text{ mm/s} \qquad v_{drift} = 1\text{ mm/s (15 A)}$$

Lights turn on immediately, why?

Origin of Ohm's law

Stationary charges ---> electrostatics

Constant current ---> magnetostatics

$\left(\begin{array}{c} \text{Classical calculations are} \\ \text{within } \pm 1 \text{ order of mag.} \end{array} \right)$

What is the origin of Ohm's law, $V = IR$?

Key is thermal velocity v_{th} of electrons is >> than 1 mm/s.

$\left(\begin{array}{c} \text{QM description} \\ \text{required here} \end{array} \right)$

Time between collisions τ involves mean the free path $\langle \lambda \rangle$

$$\tau = \langle \lambda \rangle / v_{th}$$

Electron drift velocity is then

$$v_{drift} = a\tau = \frac{qE}{m}\frac{\langle \lambda \rangle}{v_{th}} = \frac{qV}{mL}\frac{\langle \lambda \rangle}{v_{th}}$$

Current from previous slide

$$I = qn_L v_{drift} = \frac{q^2 n_L}{mL}\frac{\langle \lambda \rangle}{v_{th}} V \qquad I = \frac{V}{R}$$

Resistance per unit length

$$\frac{R}{L} = \frac{mv_{th}}{q^2 n_L \langle \lambda \rangle}$$

Resistivity ρ_R

$$\rho_R = \frac{L}{RA}$$

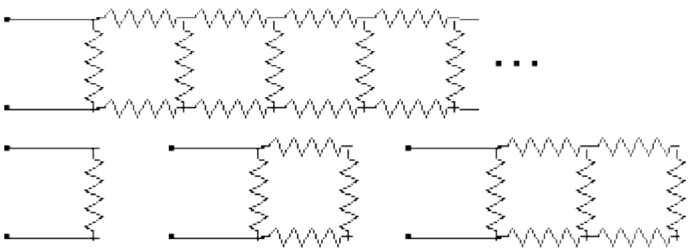
Conductivity σ_R (beware, σ also surface Q density)

$$\sigma_R = \frac{L}{RA} = \frac{n_L}{A}\frac{q^2 \langle \lambda \rangle}{mv_{th}}$$

$$\sigma_R = \frac{n_V q^2 \langle \lambda \rangle}{mv_{th}}$$

Resistors

Each resistor has the value R. What is the total resistance, R_T, of this infinite set of resistors?

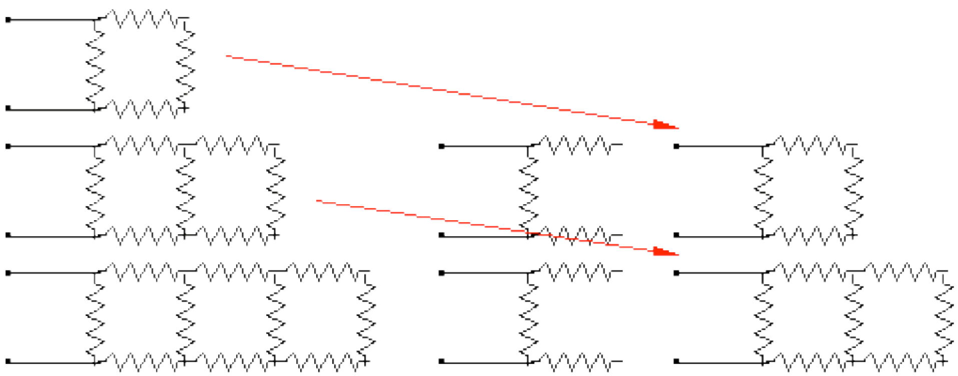


One resistor: $R_T = R$

Four resistors: $R_T = R +$ (3R in parallel)

N resistors: $R_T =$ not obvious

The trick!



Current and current densities

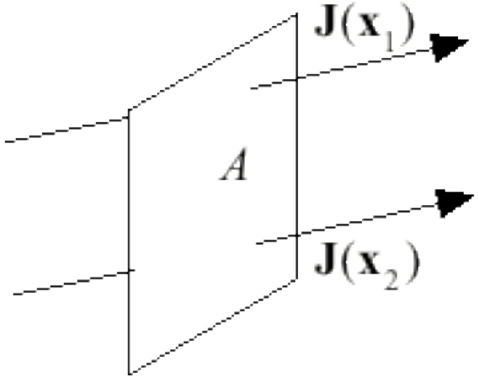
No charge can accumulate in a uniform wire carrying current. Charge going in = charge going out.

$\rho_c(\mathbf{x}) = \text{charge density}$ $\qquad \dfrac{\partial \rho_c}{\partial t} = 0$

Beware: ρ is also used for resistivity of a material.

Volume current density $\mathbf{J}(\mathbf{x}) = n_V q \mathbf{v}(\mathbf{x})$ (3 D)

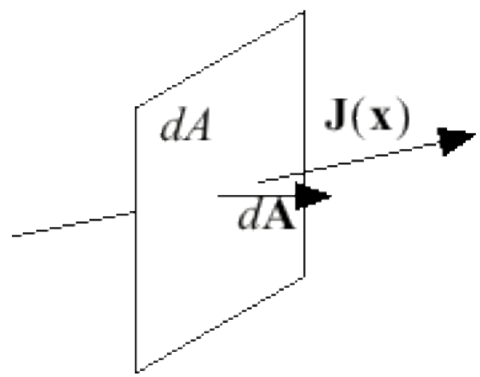
Current density **J** is position dependent



Current is charge through A per unit time

$I = dQ/dt$ (1 D)

Local current density **J** for a differential area element

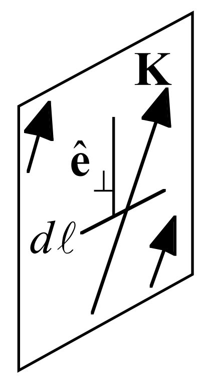


Current through *dA*

$dI = \mathbf{J}(\mathbf{x}) \cdot d\mathbf{A}$ (1 D)

$\mathbf{K}(\mathbf{x}) = n_S q \mathbf{v}(\mathbf{x})$ (2 D)

Surface current **K**



Current crossing $d\ell$

$dI = \mathbf{K} \cdot \hat{\mathbf{e}}_\perp d\ell$ (1 D)

Continuity equation -- conservation of charge

Remember that ρ_c and **J** are functions of 3D position

$$\rho_c(\mathbf{x}) \ \& \ \mathbf{J}(\mathbf{x})$$

Continuity equation: $$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_c}{\partial t}$$

Integrate over volume V, and use Gauss's theorem

$$\int_V \nabla \cdot \mathbf{J} \, d^3x = -\int_V \frac{\partial \rho_c}{\partial t} \, d^3x$$

Rate of change of charge inside of V

Continuity equation, integral form.

Flux of **J** through surface S

$$\int_S \mathbf{J} \cdot d\mathbf{A} = -\frac{d}{dt} \int_V \rho_c d^3x$$

Rate of change of charge inside of V

Boundary condition on **J** at a surface

Discontinuity in normal component of **J** at a surface

$$J_{2n} - J_{1n} = -\frac{\partial \sigma_c}{\partial t}$$

Rate of change of surface charge density

$\frac{\partial \rho_c}{\partial t} = 0$ and $\frac{\partial \sigma_c}{\partial t} = 0$ with constant currents

Reminiscent of $$E_{2n} - E_{1n} = \frac{\sigma_c}{\varepsilon_0}$$

Conductors

Macroscopic | Microscopic - Local forms of Ohm's law

Resistance

$$I = \frac{1}{R} V$$

Resistivity ρ

$$\mathbf{J}(\mathbf{x}) = \frac{1}{\rho_R} \mathbf{E}(\mathbf{x})$$

Conductivity σ

$$\mathbf{J}(\mathbf{x}) = \sigma_R \, \mathbf{E}(\mathbf{x})$$

$$\sigma_R = \frac{1}{\rho_R}$$

ρ and σ are an intrinsic property of a material

"Wire" with area A, and resistivity ρ_R

$$R = \int dR = \frac{1}{A} \int_0^z \rho_R(z') dz'$$

With constant resistivity ρ_R

$$R = \frac{\rho_R}{A} z$$

Resistance linear in z (length) and ρ .

What is still true about **E** ?

$\mathbf{E} = -\nabla V$ field <--> potential relationship, e.g., $\mathbf{E} = E_z \hat{\mathbf{k}}; \quad V = -E_z z$

$\nabla \times \mathbf{E} = 0$ violation requires changing magnetic fields

$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\varepsilon_0} \Rightarrow \frac{\rho_{free}}{\varepsilon}$ must specify "free charge density" and permittivity, ε .

Free charge and resistivity

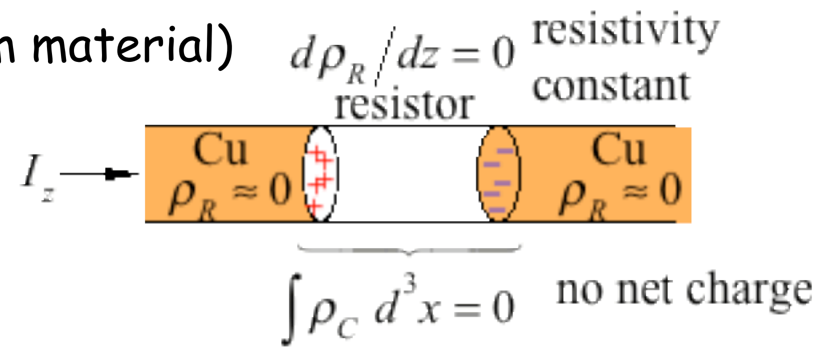
What is free charge, ρ_{free} ?

Though currents have moving charge, resistance will allow charge to accumulate locally.

$\rho_{free} = 0$ with *constant* resistivity (uniform material)

No free charge in resistor volume

Free charge on up- & down-stream surfaces



$$\rho_{free} = \frac{\varepsilon I}{A}\left(\frac{d\rho_R}{dz}\right)$$ with resistivity changing over z

Free charge in resistor volume, and

free charge on up- & down-stream surfaces