# PHY481: Electromagnetism

Homework 7

Magnetic Fields

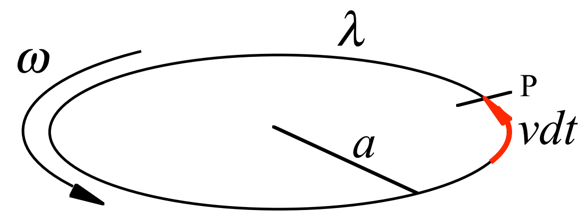
# HW 7

## Problem 7.2

$$dQ = \lambda v dt$$

$$I = \frac{dQ}{dt} = \lambda v$$

$$I = \lambda v = \lambda \omega a$$




## Problem 7.3

Quick : $\mathbf{J}(r) = qn\mathbf{v} = \rho_c \boldsymbol{\omega} \times \mathbf{r} = \rho_c \omega r \sin\theta \hat{\boldsymbol{\phi}}$

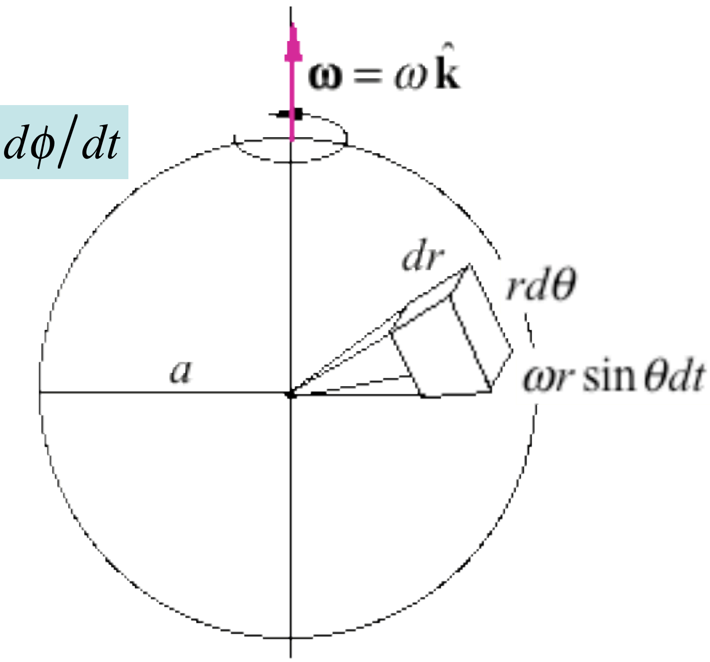
Slow but sure : $dA = rdrd\theta$ $\quad d^3x = r^2 dr \sin\theta d\theta d\phi$ $\quad \omega = d\phi/dt$

$$\int_S \mathbf{J}(r) \cdot d\mathbf{A} = \frac{d}{dt} \int_V \rho_c d^3x$$

$$\int_S J_\phi(r) rdrd\theta = \int_V \left( \rho_c r \sin\theta \frac{d\phi}{dt} \right) rdrd\theta$$

$$J_\phi(r) = \rho_c r \sin\theta \omega$$

$$\mathbf{J}(r) = \rho_c r \sin\theta \omega \hat{\boldsymbol{\phi}}$$



# HW 7 (cont'd)

## Problem 7.5

$$A = \frac{\pi}{4} D^2; \quad A' = \frac{\pi}{4}(D')^2 = \frac{\pi}{4}(D/2)^2 = \frac{A}{4}$$

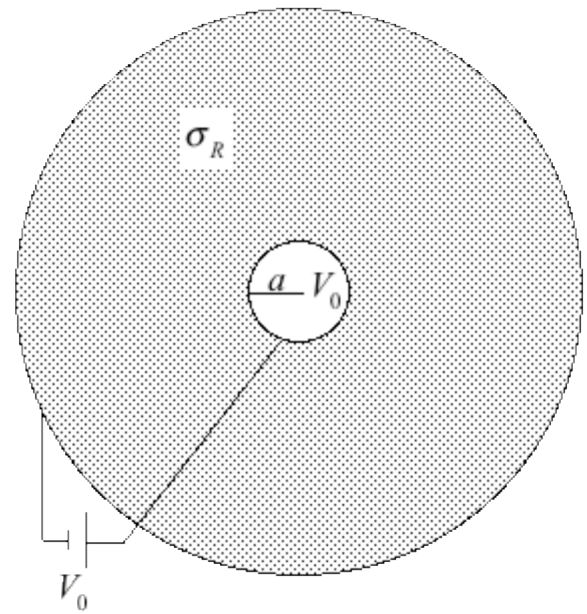
$$R' = \frac{\rho_R L'}{A'} = \frac{\rho_R 2L}{A/4} = 8\frac{\rho_R L}{A} = 8R$$

## Problem 7.7

$$V(r) = \frac{Q_0}{4\pi\varepsilon_0 r}; \qquad \frac{Q_0}{4\pi\varepsilon_0 a} = V_0 \qquad Q_0 = V_0 4\pi\varepsilon_0 a$$

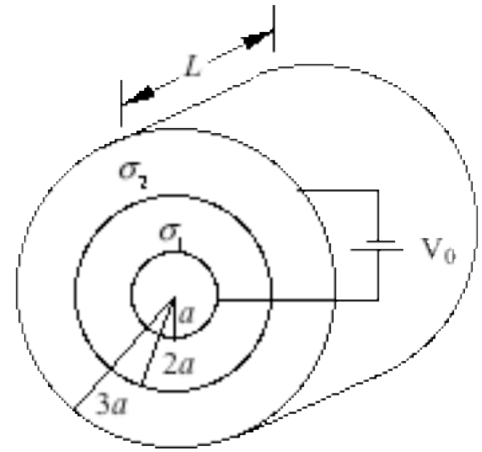
$$= \frac{V_0 a}{r}$$

$$\mathbf{E} = -\nabla V = \frac{V_0 a}{r^2}\hat{\mathbf{r}} \qquad \mathbf{J}(r) = \sigma_R \mathbf{E} = \frac{\sigma_R V_0 a}{r^2}\hat{\mathbf{r}}$$



# HW 7 (cont'd)

7.9 Cylinders length L , resistive media between radius $a$ and $3a$. What is the resistance? What is the charge density on the interface at $2a$?




General solutions

$$V_1(r) = A\ln r + B$$
$$V_2(r) = C\ln r + D$$

Current densities

$$J_1(r) = \sigma_1 E_1 = \sigma_1\left(-\nabla V_1\right) = -\sigma_1 A/r$$
$$J_2(r) = \sigma_2 E_2 = \sigma_2\left(-\nabla V_2\right) = -\sigma_2 C/r$$

Boundary Conditions

$$V_1(a) = V_0$$
$$V_2(3a) = 0$$
$$V_1(2a) = V_2(2a)$$
$$J_1(2a) = J_2(2a)$$

Determine coefficients

$$B = V_0 - A\ln a; \quad C = A\sigma_1/\sigma_2$$
$$D = -C\ln 3a = -A\sigma_1/\sigma_2 \ln 3a$$

At $r = 2a$: $A\ln(2a) + V_0 - A\ln(a) = A\ln(2a)\sigma_1/\sigma_2 - A\sigma_1/\sigma_2\ln 3a$

$$A\left[(\sigma_1/\sigma_2)\ln\left(\tfrac{2}{3}\right) + \ln\left(\tfrac{1}{2}\right)\right] = V_0 \rightarrow \underline{A = V_0/\left[(\sigma_1/\sigma_2)\ln\left(\tfrac{2}{3}\right) + \ln\left(\tfrac{1}{2}\right)\right]}$$

$$\sigma_{free} = \varepsilon_0 \Delta E = \varepsilon_0 (A - C)/2a = \frac{\varepsilon_0 V_0(\sigma_1/\sigma_2 - 1)}{2a\left[\ln\left(\tfrac{3}{2}\right) + \ln(2)\sigma_1/\sigma_2\right]}$$

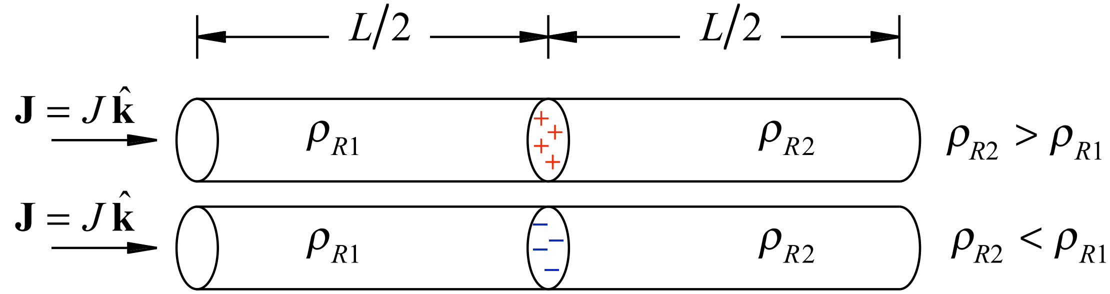
$$I = \int_{r=a} J dA = J_1(a)[2\pi a L] = \frac{2\pi L\sigma_1 V_0}{\left[(\sigma_1/\sigma_2)\ln\left(\tfrac{3}{2}\right) + \ln(2)\right]}$$

$$R = \frac{V_0}{I} = \frac{\left[(\sigma_1/\sigma_2)\ln\left(\tfrac{3}{2}\right) + \ln(2)\right]}{2\pi L\sigma_1}$$

# HW 7 (cont'd)

## Problem 7.11

$$E_1 = J\rho_{R1}; \quad E_2 = J\rho_{R2}$$




$$\sigma_c^{free} = \varepsilon_0\left(E_2 - E_1\right) = \varepsilon_0 J\left(\rho_{R2} - \rho_{R1}\right)$$

$$\sigma_c^{free} = \varepsilon_0 \frac{I\left(\rho_{R2} - \rho_{R1}\right)}{A}$$

## Problem 7.15

$$Q(t) = Q_0 e^{-t/\tau} = Q_0 e^{-t/RC}$$

$$Q(\tau)/Q_0 = Q(2\tau)/Q(\tau) = Q(6\tau)/Q(5\tau) = e^{-1}$$

$$f(t) = Q(t)/Q_0 = e^{-t/\tau}; \quad t = -\tau \ln f$$

$$t_{f=0.1} = -\tau \ln(0.1) = 2.3\tau$$

Change in stored energy in time *t*:

$$\Delta\varepsilon(t) = \frac{1}{2}\left(QV - Q_0V_0\right) = \frac{Q^2(t) - Q_0^2}{2C} = \frac{Q_0^2\left(e^{-2t/RC} - 1\right)}{2C}$$

zero sum

Energy dissipation from resistor in time t:

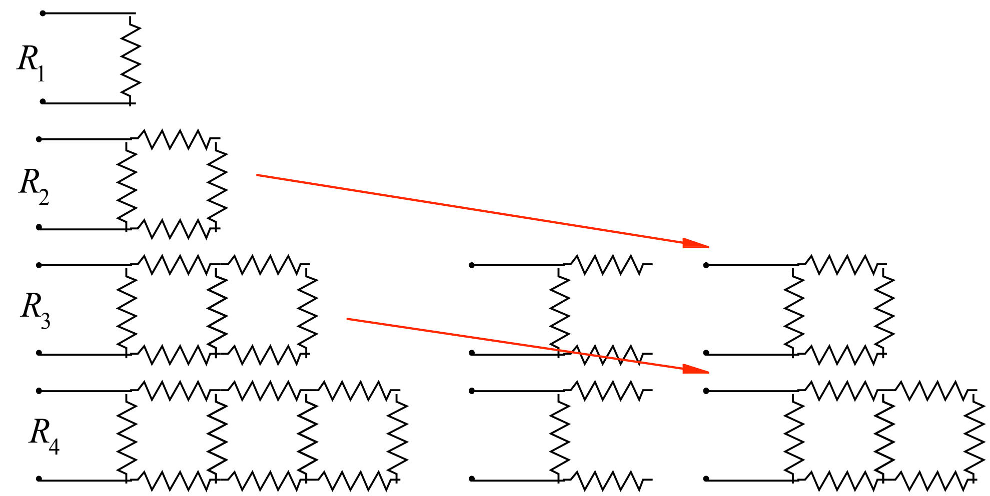
$$I(t) = \frac{dQ(t)}{dt} = -\frac{Q(t)}{RC}$$

$$\varepsilon_R(t) = \int_0^t dt P(t) = \int_0^t dt (I^2 R) = \frac{1}{RC^2}\int_0^t dt Q^2(t) = \frac{Q_0^2}{2C}\left[1 - e^{-2t/RC}\right]$$

# HW 7 (cont'd)

Problem 7.18

$$R_1 = R$$
$$1/R_2 = 1/R + 1/(2R + R_1)$$
$$1/R_3 = 1/R + 1/(2R + R_2)$$
$$1/R_n = 1/R + 1/(2R + R_{n-1})$$




$$R_n = \frac{R(2R + R_{n-1})}{3R + R_{n-1}}$$
$$R_\infty = \frac{R(2R + R_\infty)}{3R + R_\infty}$$

$$R_\infty^2 + 2RR_\infty - 2R^2 = 0$$
$$R_\infty = R(\sqrt{3} - 1)$$

# Magnetic force and field

Magnetic force: $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$

Perpendicular to direction of motion

Magnetic forces do no work $\Delta KE = 0$

Circular motion: $\mathbf{v}_0 \perp \mathbf{B}$

Centripetal acceleration caused by magnetic force. $\omega = v/R = qB/m$

$\mathbf{v}_0 \not\perp \mathbf{B}$

Helical motion. Helix has radius and pitch

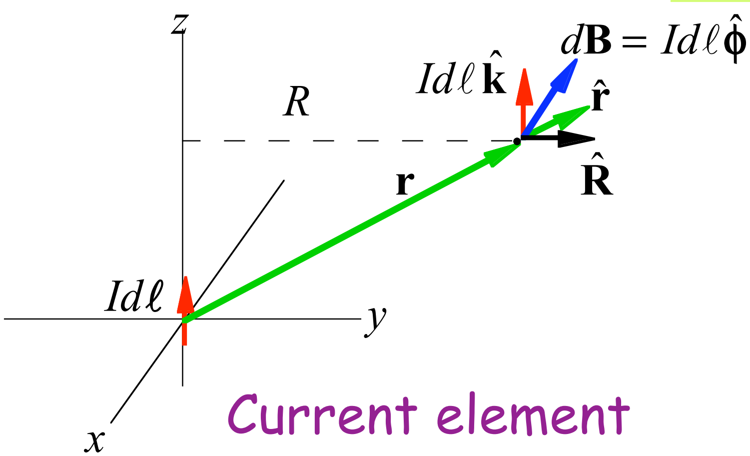
Crossed E and B: $\mathbf{E} \perp \mathbf{B}$

Exists an un-deflected velocity $\mathbf{v} = \mathbf{E} \times \mathbf{B} / B^2$

Currents make magnetic fields: Biot-Savart Law

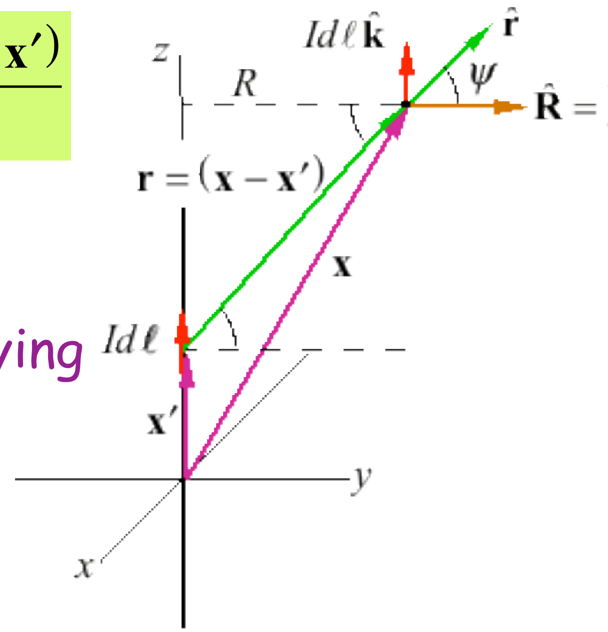
$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I d\boldsymbol{\ell} \times \hat{\mathbf{r}}}{r^2}$$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_{wire} \frac{I d\boldsymbol{\ell} \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}$$




Current element

Wire carrying current



# Magnetic field of long wire carrying current

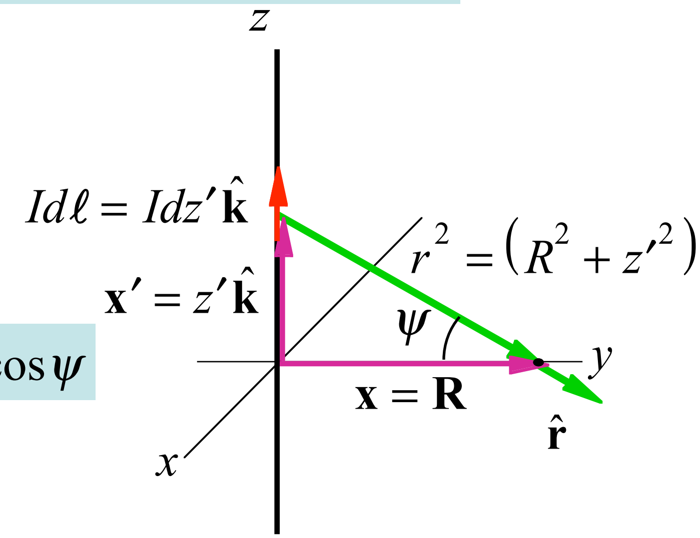
## (the hard way, later we will use Ampere's law)

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_{wire}\frac{I d\ell \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}$$

$$\mathbf{x} = R\hat{\mathbf{R}} + z\hat{\mathbf{k}} \qquad \mathbf{x}' = z'\hat{\mathbf{k}} \qquad d\ell = dz'\hat{\mathbf{k}}$$

Magnetic field does not depend on z.

Any z will do. Determine B-field at z = 0




$$\hat{\mathbf{r}} = \hat{\mathbf{R}}\cos\psi - \hat{\mathbf{k}}\sin\psi$$

$$\hat{\mathbf{k}}\times\hat{\mathbf{r}} = \hat{\mathbf{k}}\times\hat{\mathbf{R}}\cos\psi = \hat{\boldsymbol{\phi}}\cos\psi$$

$$\cos\psi = R\Big/\left(R^2 + z'^2\right)^{1/2}$$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty}\frac{dz'\,\hat{\mathbf{k}}\times\hat{\mathbf{r}}}{\left(R^2 + z'^2\right)}$$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I R}{4\pi}\int_{-\infty}^{\infty}\frac{dz'}{\left(R^2 + z'^2\right)^{3/2}}\hat{\boldsymbol{\phi}}$$

$$\int_{-\infty}^{\infty}\frac{dz'}{\left(R^2 + z'^2\right)^{3/2}} = \left[\frac{z'}{R^2\sqrt{R^2 + z'^2}}\right]_{-\infty}^{\infty} = \frac{2}{R^2}$$

$$\mathbf{B}(R) = \frac{\mu_0 I}{2\pi R}\hat{\boldsymbol{\phi}}$$