PHY481: Electromagnetism

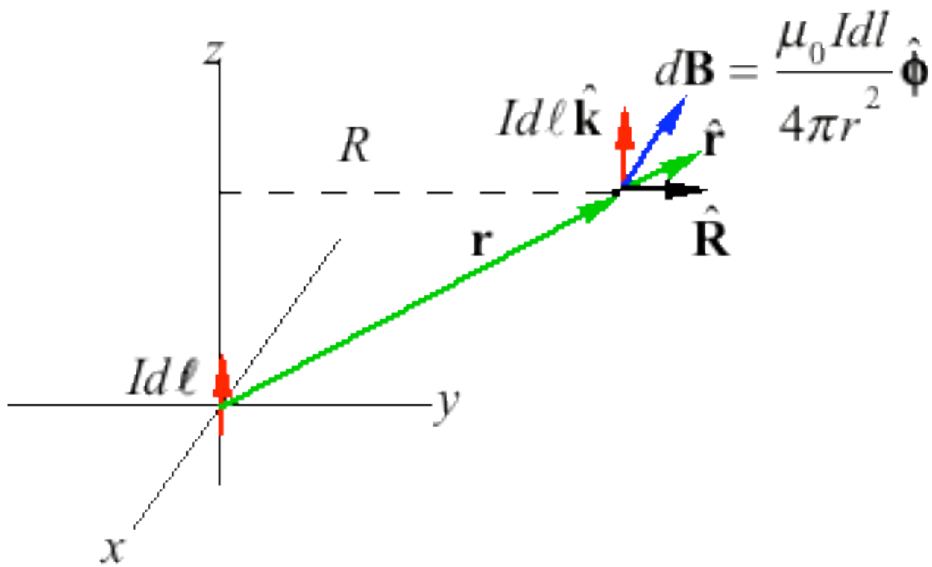
Magnetic Fields

Magnetic field

Currents make magnetic fields: Biot-Savart Law

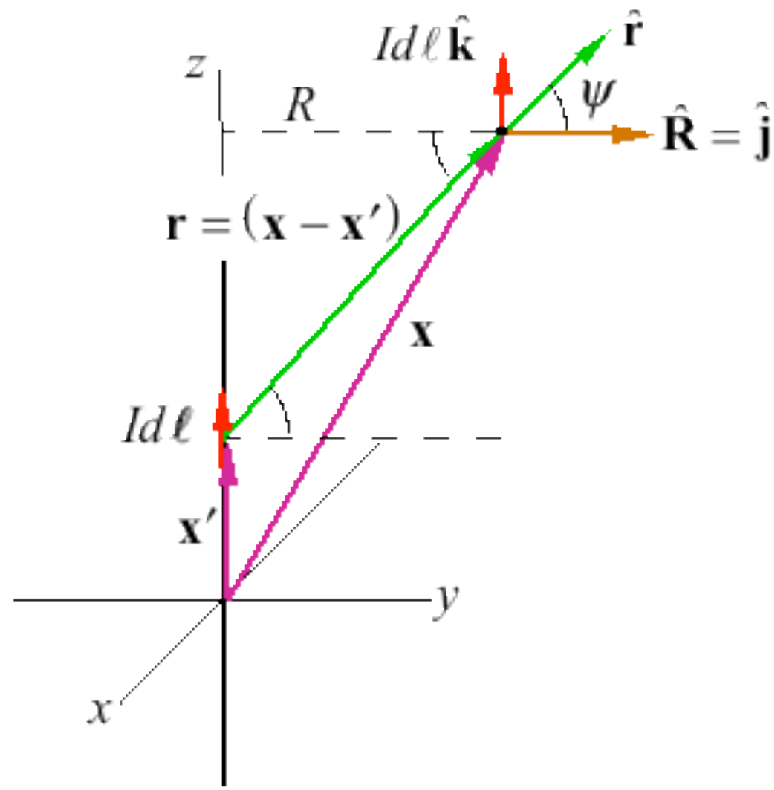
Current element

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{Id\ell \times \hat{\mathbf{r}}}{r^2}$$



Straight piece of wire carrying current

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_{wire}\frac{Id\ell \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}$$



Magnetic field of long wire carrying current

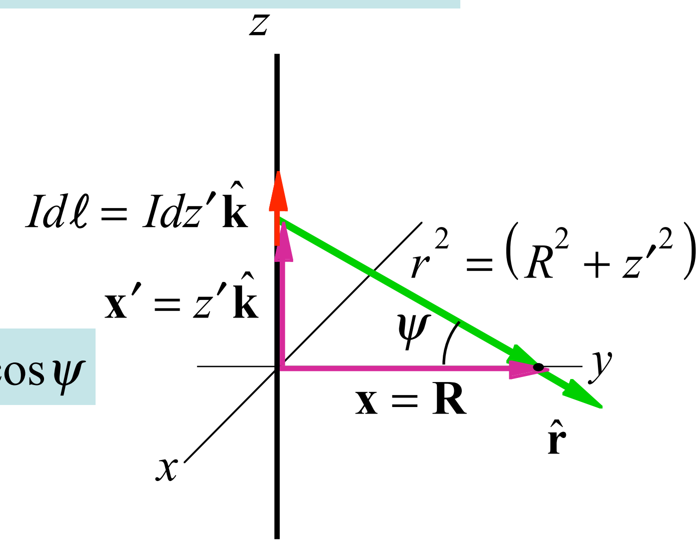
(the hard way, later we will use Ampere's law)

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_{wire} \frac{I d\boldsymbol{\ell} \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}$$

$$\mathbf{x} = R\hat{\mathbf{R}} + z\hat{\mathbf{k}} \qquad \mathbf{x}' = z'\hat{\mathbf{k}} \qquad d\boldsymbol{\ell} = dz'\hat{\mathbf{k}}$$

Magnetic field does not depend on z.

Any z will do. Determine B-field at z = 0



$$\hat{\mathbf{r}} = \hat{\mathbf{R}}\cos\psi - \hat{\mathbf{k}}\sin\psi$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{r}} = \hat{\mathbf{k}} \times \hat{\mathbf{R}}\cos\psi = \hat{\boldsymbol{\phi}}\cos\psi$$

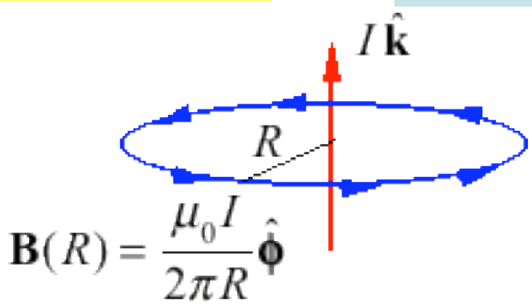
$$\cos\psi = R\big/\left(R^2 + z'^2\right)^{1/2}$$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty} \frac{dz'\,\hat{\mathbf{k}} \times \hat{\mathbf{r}}}{\left(R^2 + z'^2\right)}$$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I R}{4\pi}\int_{-\infty}^{\infty} \frac{dz'}{\left(R^2 + z'^2\right)^{3/2}}\hat{\boldsymbol{\phi}}$$

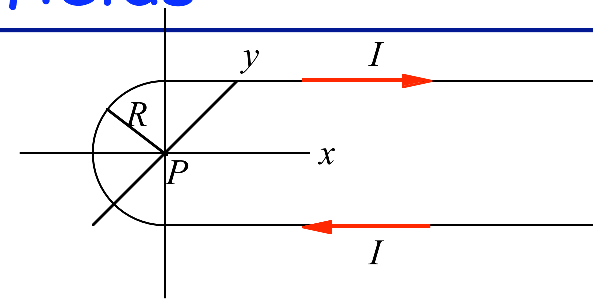
$$\int_{-\infty}^{\infty} \frac{dz'}{\left(R^2 + z'^2\right)^{3/2}} = \left[\frac{z'}{R^2\sqrt{R^2 + z'^2}}\right]_{-\infty}^{\infty} = \frac{2}{R^2}$$

$$\mathbf{B}(R) = \frac{\mu_0 I}{2\pi R}\hat{\boldsymbol{\phi}}$$



More magnetic fields

Field at center of a narrow loop
= 2 straight wires + 1/2 ring



Field of long straight wire

$$\mathbf{B} = \frac{\mu_0 I}{2\pi R}\hat{\phi}$$

end of long wire

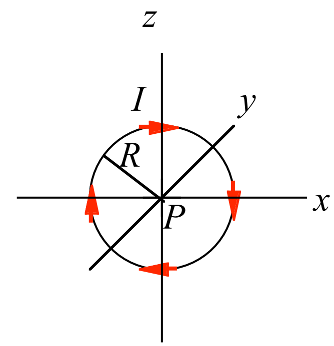
$$\mathbf{B} = \frac{\mu_0 I}{4\pi R}\hat{\phi}$$

Field at center of a current ring

$$\mathbf{B} = \int d\mathbf{B} = \frac{\mu_0}{4\pi}\int \frac{I d\ell \times (-R\hat{\mathbf{r}})}{R^3}$$

$$= \hat{\mathbf{j}}\frac{\mu_0 I}{4\pi R^2}\int_0^{2\pi} R d\phi = \hat{\mathbf{j}}\frac{\mu_0 I}{2R}$$

$-R\hat{\mathbf{r}}$ because $\mathbf{r}$ points from d$\mathbf{l}$ to P



Field at center of 1/2 a current ring

$$\mathbf{B} = \frac{\mu_0 I}{4R}$$

Field at center of narrow loop

$$\mathbf{B} = \hat{\mathbf{j}}\frac{\mu_0 I}{2R}\left(\frac{1}{2}+\frac{1}{\pi}\right) \qquad \mathbf{B} = \hat{\mathbf{j}}\frac{\mu_0 I}{4\pi R}(\pi+2)$$

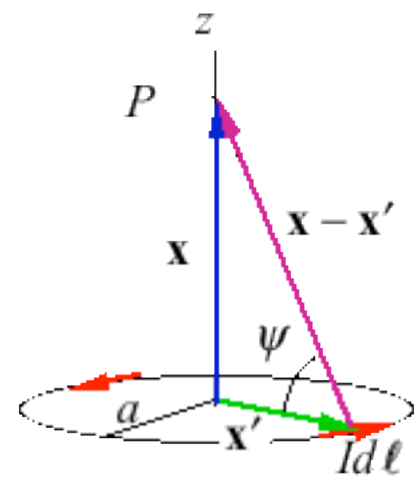
More magnetic fields

Field on axis of current loop

$$\mathbf{x} = z\hat{\mathbf{k}}; \quad \mathbf{x}' = R\hat{\mathbf{R}}; \quad Id\ell = ad\phi\,\hat{\boldsymbol{\phi}}$$

$$\mathbf{r} = \mathbf{x} - \mathbf{x}' = z\hat{\mathbf{k}} - a\hat{\mathbf{R}}$$

$$r\,\hat{\mathbf{r}} = \sqrt{a^2 + z^2}\left(\hat{\mathbf{k}}\sin\psi - \hat{\mathbf{R}}\cos\psi\right)$$



$$\hat{\boldsymbol{\phi}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \times \hat{\mathbf{k}}\sin\psi - \hat{\boldsymbol{\phi}} \times \hat{\mathbf{R}}\cos\psi$$

$$= +\hat{\mathbf{R}}\sin\psi + \hat{\mathbf{k}}\cos\psi$$

$$\cos\psi = a\Big/\sqrt{a^2 + z^2}$$

Component in **R** direction will cancel in the integral over phi

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{Id\ell \times (\mathbf{x} - \mathbf{x}')\,d^3x'}{|\mathbf{x} - \mathbf{x}'|^3}$$

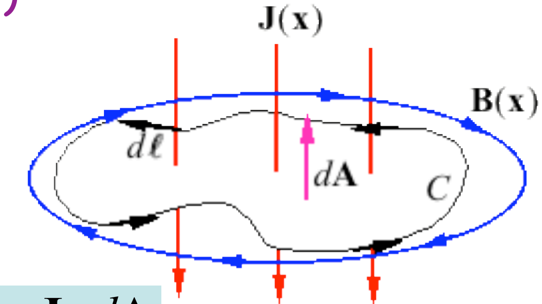
$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I a^2}{4\pi}\int \frac{d\phi\,\hat{\mathbf{k}}}{\left(a^2 + z^2\right)^{3/2}}$$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I a^2}{2\left(a^2 + z^2\right)^{3/2}}\hat{\mathbf{k}}$$

Ampere's Law

$\nabla \cdot \mathbf{B} = 0$ Magnetic fields are "solenoidal" (always true)

$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ Ampere's Law (constant currents only. Maxwell's equations have another term)



Integrate over a surface element: $\oint_S (\nabla \times \mathbf{B}) \cdot \mathbf{n}\, dA = \mu_0 \oint_S \mathbf{J} \cdot d\mathbf{A}$

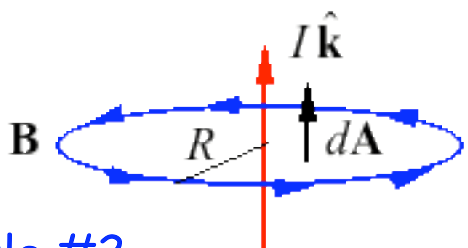
Apply Stokes's theorem: $\oint_C \mathbf{B} \cdot d\ell = \mu_0 \oint_S \mathbf{J} \cdot d\mathbf{A}$

$dI = \mathbf{J} \cdot d\mathbf{A}$

$$\oint_C \mathbf{B} \cdot d\ell = \mu_0 I_{encl}$$

Ampere's Law and symmetries

Long straight wire (the easy way)



$$\oint_C \mathbf{B} \cdot d\ell = \mu_0 I_{encl}$$

$$B_\phi 2\pi R = \mu_0 I$$

$$\mathbf{B}(R) = \frac{\mu_0 I}{2\pi R} \hat{\phi}$$

Right hand rule #2

Ampere's law applications

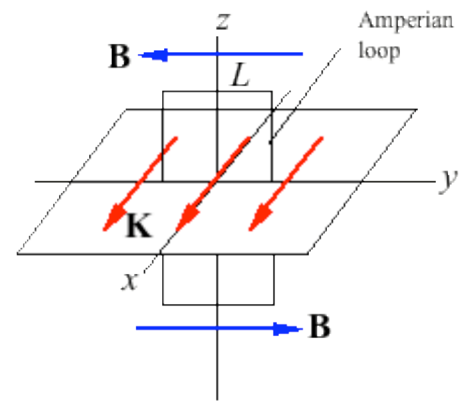
Each line of current creates a B field in –y direction above the plane and +y direction below.

Infinite current sheet

$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ $\quad \oint \mathbf{B} \cdot d\ell = \mu_0 I_{encl}$ $\quad I_{encl} = \mathbf{K} \cdot \hat{\mathbf{i}} L = KL$

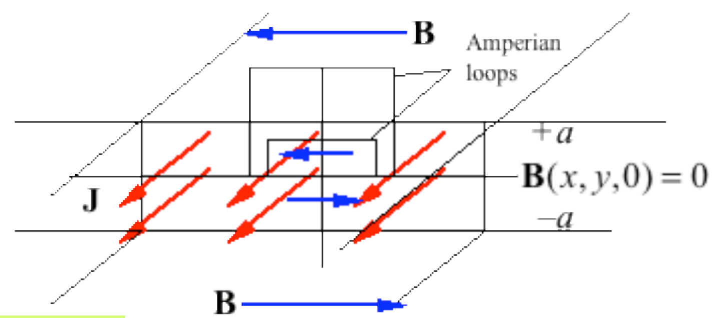
$$2BL = \mu_0 KL; \quad B = \frac{\mu_0 K}{2}$$

$$\mathbf{B} = -\frac{\mu_0 K}{2}\hat{\mathbf{j}} \quad z > 0; \quad \mathbf{B} = +\frac{\mu_0 K}{2}\hat{\mathbf{j}} \quad z > 0$$



B field on the center line must be zero

Infinite current slab



$$\oint \mathbf{B} \cdot d\ell = \mu_0 I_{encl}$$

Inside

$$I_{encl} = \int \mathbf{J} \cdot d\mathbf{A} = JLz$$

$$\oint \mathbf{B} \cdot d\ell = (-B\hat{\mathbf{j}})(-L\hat{\mathbf{j}}) = BL$$

$$\mathbf{B}_{top} = -\mu_0 Jz\,\hat{\mathbf{j}} \quad \mathbf{B}_{bottom} = \mu_0 Jz\,\hat{\mathbf{j}}$$

Outside

$$I_{encl} = \int \mathbf{J} \cdot d\mathbf{A} = JLa$$

$$\oint \mathbf{B} \cdot d\ell = BL$$

$$\mathbf{B}_{top} = -\mu_0 Ja\,\hat{\mathbf{j}} \quad \mathbf{B}_{bottom} = \mu_0 Ja\,\hat{\mathbf{j}}$$

Field equations for B(x) & Vector potential A(x)

Always true | Constant currents only.

Constant currents:

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \cdot (\nabla \times \mathbf{B}) = 0 = \mu_0 \nabla \cdot \mathbf{J}$$

$$\nabla \cdot \mathbf{J} = 0$$

Ampere's Law:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\oint_C \mathbf{B} \cdot d\ell = \mu_0 I_{encl}$$

Biot-Savart Law:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}$$

$$\frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} = -\nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|}$$

$$\mathbf{B}(\mathbf{x}) = \nabla \times \left[\frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') d^3x'}{|\mathbf{x} - \mathbf{x}'|} \right]$$

$$\left(\nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \times \mathbf{J}(\mathbf{x}') = \nabla \times \left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right)$$

$$\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x})$$

Fields from potentials

Vector potential:

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') d^3x'}{|\mathbf{x} - \mathbf{x}'|}$$

$$\mathbf{E} = -\nabla V$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Vector potential calculations

If currents are bounded (do not extend to infinity) get A by

Volume currents

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

Surface currents

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{x}')d^2x'}{|\mathbf{x}-\mathbf{x}'|}$$

Line currents

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{I d\ell}{|\mathbf{x}-\mathbf{x}'|}$$

If currents extend to infinity get A via Stokes's theorem and B:

$$\oint_C \mathbf{A}\cdot d\ell = \int (\nabla\times\mathbf{A})\cdot d\mathbf{a} = \int \mathbf{B}\cdot d\mathbf{a}$$