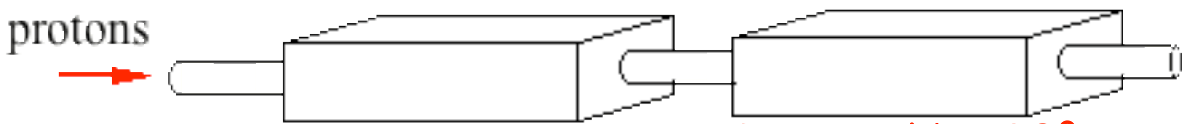
PHY481: Electromagnetism

Quadrupole doublet

Vector Potential

The quadrupole doublet

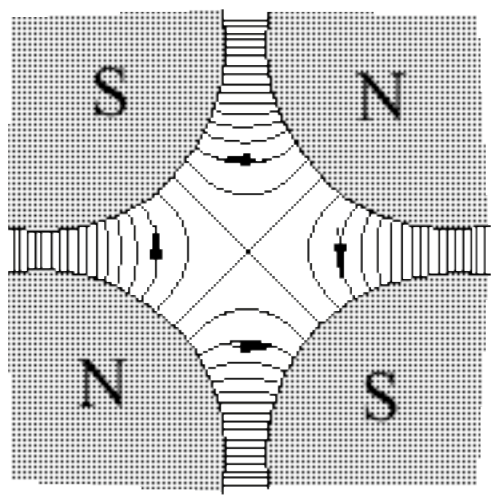
The quadrupole doublet. Focuses (weakly) in both dimensions

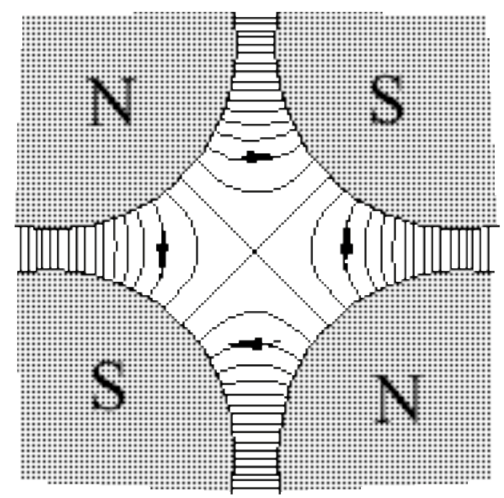


Rotated by 90°

left or right, focused
top or bottom, defocused

left or right, defocused
top or bottom, focused





left or right, closer inward
top or bottom, further out

Field equations for B(x) & Vector potential A(x)

Always true | Constant currents only.

Constant currents:

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \cdot (\nabla \times \mathbf{B}) = 0 = \mu_0 \nabla \cdot \mathbf{J}$$

$$\nabla \cdot \mathbf{J} = 0$$

Ampere's Law:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\oint_C \mathbf{B} \cdot d\ell = \mu_0 I_{encl}$$

Biot-Savart Law:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}') \, d^3x'}{|\mathbf{x} - \mathbf{x}'|^3}$$

$$\frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} = -\nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|}$$

$$\mathbf{B}(\mathbf{x}) = \nabla \times \left[\frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') d^3x'}{|\mathbf{x} - \mathbf{x}'|} \right]$$

$$\left(\nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \times \mathbf{J}(\mathbf{x}') = \nabla \times \left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right)$$

$$\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x})$$

Fields from potentials

$$\mathbf{E} = -\nabla V$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Vector potential:

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') d^3x'}{|\mathbf{x} - \mathbf{x}'|}$$

Identity for previous slide

Prove $$\left(\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\times\mathbf{J}(\mathbf{x}')=\nabla\times\left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right)$$

$$\text{Let } f(\mathbf{x})=\frac{1}{|\mathbf{x}-\mathbf{x}'|}$$

$$[\nabla f(\mathbf{x})]_j=\frac{\partial f(\mathbf{x})}{\partial x_j}$$

$$\begin{aligned}[\nabla f(\mathbf{x})]\times\mathbf{J}(\mathbf{x}') &= \varepsilon_{ijk}[\nabla f(\mathbf{x})]_j J_k(\mathbf{x}')\\ &= \varepsilon_{ijk}\frac{\partial f(\mathbf{x})}{\partial x_j}J_k(\mathbf{x}')=\varepsilon_{ijk}\frac{\partial}{\partial x_j}[f(\mathbf{x})J_k(\mathbf{x}')]\\ &= \nabla\times[f(\mathbf{x})\,\mathbf{J}(\mathbf{x}')]\end{aligned}$$

and

$$\mathbf{J}(x')\times[\nabla f(x)]=-\nabla\times[f(x)\,\mathbf{J}(x')]$$

Vector potential calculations

If currents are bounded (do not extend to infinity) get A by

Volume currents

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

Surface currents

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{x}')d^2x'}{|\mathbf{x}-\mathbf{x}'|}$$

Line currents

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{I d\ell}{|\mathbf{x}-\mathbf{x}'|}$$

If currents extend to infinity get A via Stokes's theorem and B:

$$\oint_C \mathbf{A} \cdot d\ell = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int \mathbf{B} \cdot d\mathbf{a}$$

Direction of ***A*** can be tricky

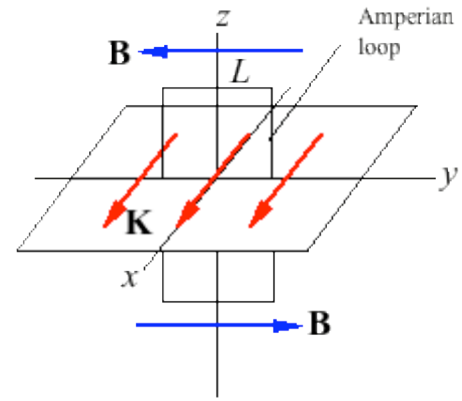
Infinite current sheet

Magnetic field B is easy:

Ampere's law around the loop shown

$$\oint \mathbf{B}\cdot d\ell = \mu_0 I_{encl} \qquad 2BL = \mu_0 KL \qquad B = \mu_0 K/2$$

$$\mathbf{B} = \frac{\mu_0 K}{2}\begin{cases} -\hat{\mathbf{j}} & z>0 \\ +\hat{\mathbf{j}} & z<0 \end{cases}$$



Vector potential A:

For currents of infinite extent

$$\oint_C \mathbf{A}\cdot d\ell = \int \mathbf{B}\cdot d\mathbf{a}$$

For bounded currents

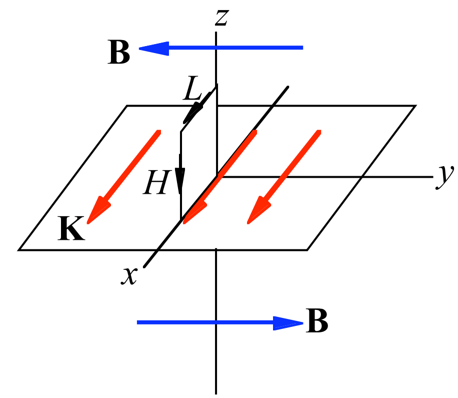
$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

"Real" currents form loops

Assume $\mathbf{A}(x,z) = f(z)\hat{\mathbf{i}} + g(x)\hat{\mathbf{k}}$

Choose A•dl loop with normal in B direction

$$\oint_C \mathbf{A}\cdot d\ell = f(H)\int_0^L dx + g(L)\int_H^0 dz = Lf(H) - Hg(L) \qquad \int \mathbf{B}\cdot d\mathbf{a} = -BHL$$



Find functions *f* and *g* such that: $Lf(H) - Hg(L) = -BHL$

$z>0$

$$f(z) = -Bz/2; g(x) = Bx/2$$

$$\mathbf{A}(x,z) = \frac{B}{2}\left(-z\,\hat{\mathbf{i}} + x\,\hat{\mathbf{k}}\right)$$

$z<0$

$$f(z) = +Bz/2; g(x) = -Bx/2$$

$$\mathbf{A}(x,z) = \frac{B}{2}\left(+z\,\hat{\mathbf{i}} - x\,\hat{\mathbf{k}}\right)$$

$\mathbf{B} = \nabla\times\mathbf{A}$ Yes!

Assumption from previous slide

$\mathbf{B} = \nabla \times \mathbf{A}$ Assume $\mathbf{A}(x,z) = f(z)\hat{\mathbf{i}} + g(x)\hat{\mathbf{k}}$

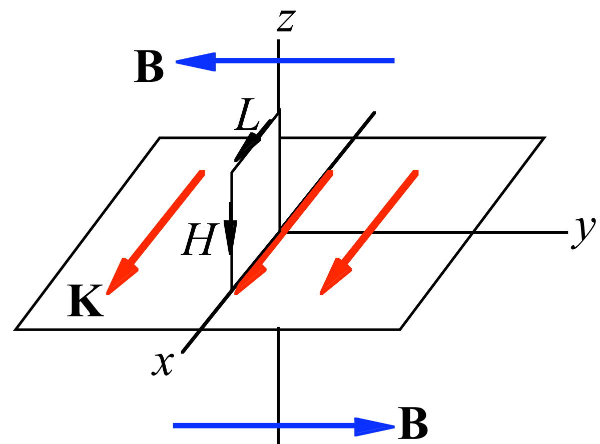
Magnetic field **B** is only in y direction

$$B_2 = \varepsilon_{231}\frac{\partial A_1}{\partial x_3} - \varepsilon_{231}\frac{\partial A_3}{\partial x_1}$$

Involves *z* dependence of A_x and, x dependence of A_z .

Visualizing the vector potential

Infinite sheet of current



Magnetic field

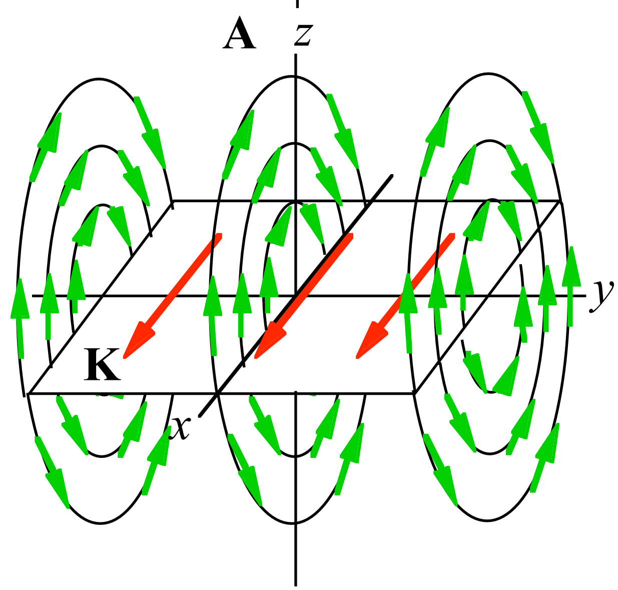
$$\mathbf{B} = \frac{\mu_0 K}{2}\begin{cases} -\hat{\mathbf{j}} & z>0 \\ +\hat{\mathbf{j}} & z<0 \end{cases}$$

Changes sign above and below the sheet

Vector potential

$$\mathbf{A} = \mu_0 K\left(-z\,\hat{\mathbf{i}} + x\,\hat{\mathbf{k}}\right)/4 \quad z>0$$
$$\mathbf{A} = \mu_0 K\left(+z\,\hat{\mathbf{i}} - x\,\hat{\mathbf{k}}\right)/4 \quad z<0$$

A_z is discontinuous across the current sheet.



Why does **A** point opposite to **K** and curve this way?

Solenoid field and vector potential

Magnetic field

Ampere's law gives

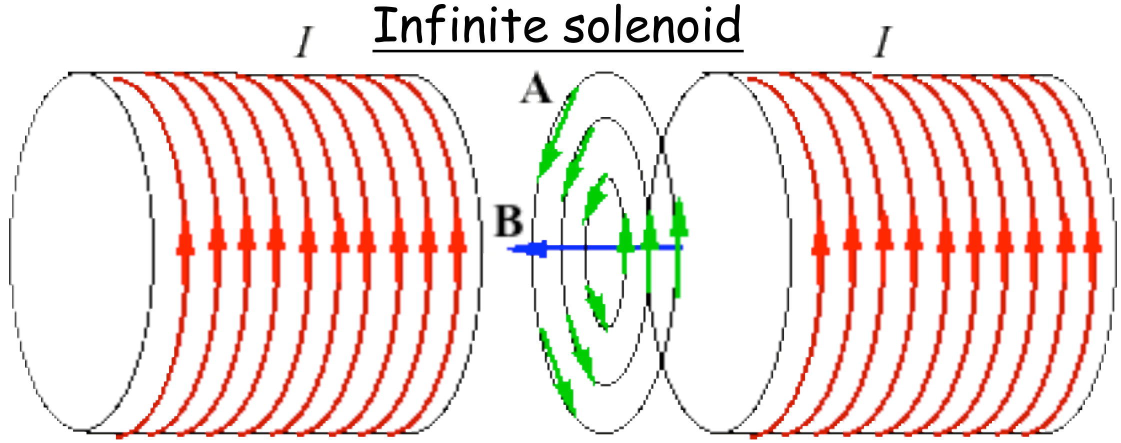
$$\mathbf{B} = -B\,\hat{\mathbf{j}} = -\mu_0 n I\,\hat{\mathbf{j}}$$

$$n = \text{turns/unit length}$$

Vector potential

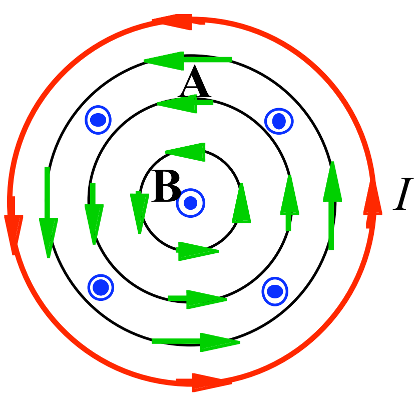
$$\mathbf{A} = B\left(-z\,\hat{\mathbf{i}} + x\,\hat{\mathbf{k}}\right)/2$$

Infinite solenoid

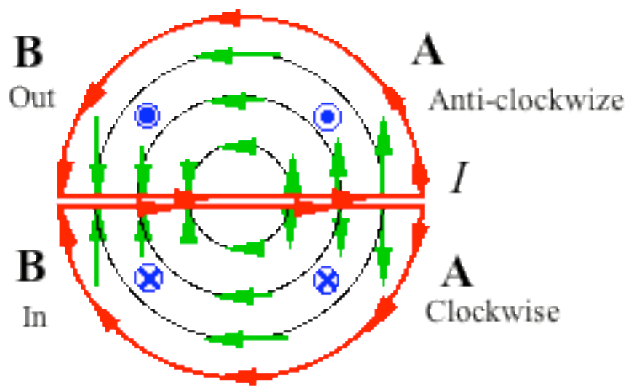


Cut-away drawing showing the axial magnetic field **B** (uniform), and *growing* circular vector potential ***A***.

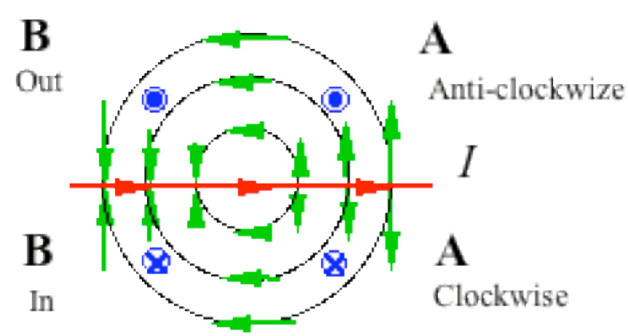
Full solenoid



Two half solenoids



Current sheet



End views. Note: **B** and ***A*** directions and compare with a current sheet.