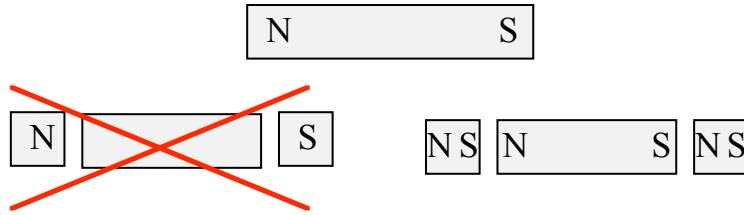

PHY481: Electromagnetism

Magnetic dipole

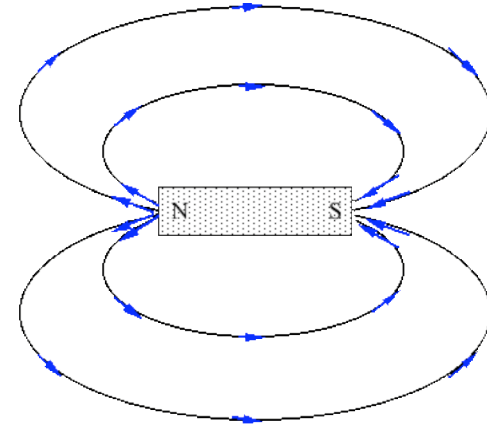
Magnetic dipole

In the many attempts to find a **magnetic monopole**, no one as ever seen one.



$$\nabla \cdot \mathbf{B} = 0 \Rightarrow \text{no magnetic monopole}$$

Maybe there is only one in the universe (Dirac and monopoles)



The vector potential of an arbitrary current distribution can tough to calculate.

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') d^3 x'}{|\mathbf{x} - \mathbf{x}'|}$$

However it can be described in a "multipole" expansion.

Legendre polynomials

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{\sqrt{r^2 - 2rr' \cos \theta + r'^2}} = \frac{1}{r} + \frac{r' \cos \theta}{r^2} + \frac{(r')^2}{r^3} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) + \dots$$

$$P_1(\cos \theta) = \cos \theta$$

$$P_2(\cos \theta) = \frac{3}{2} \cos^2 \theta - \frac{1}{2}$$

Also used for electric "multipole" expansion earlier in semester

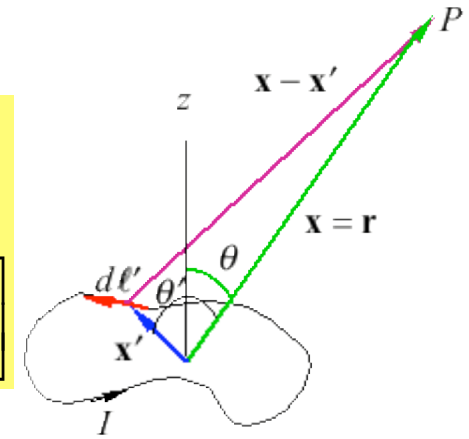
Magnetic multipole expansion

Vector potential of a thin current loop

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') d^3 x'}{|\mathbf{x} - \mathbf{x}'|}$$

$$= \frac{\mu_0 I}{4\pi r} \left[\int d\ell' + \frac{1}{r} \int r' P_1(\cos \theta') d\ell' + \frac{1}{r^2} \int (r')^2 P_2(\cos \theta') d\ell' + \dots \right]$$

monopole=0 dipole quadrupole



If this loop lies in x-y plane

ψ' is angle between $\mathbf{r}'$ and $d\ell'$

$\hat{\mathbf{n}}$ is normal to the plane

$$I \int r' \cos \theta' d\ell' = \left[\frac{I}{2} \int r' \sin \psi' d\ell' \right] \hat{\mathbf{n}} \times \hat{\mathbf{r}}$$

scalars
(mathematical wizardry)

Dipole moment

$$m = \left[\frac{I}{2} \int r' \sin \psi' d\ell' \right] \quad \mathbf{m} = m \hat{\mathbf{n}}$$

Not dependent on P

Far from the loop

$$\hat{\mathbf{n}} = \hat{\mathbf{k}}$$

$$\mathbf{A}(r, \theta) \approx \frac{\mu_0 I}{4\pi r^2} \int r' \cos \theta' d\ell' = \frac{\mu_0 m \sin \theta}{4\pi r^2} \hat{\boldsymbol{\phi}}$$

$$\mathbf{B}(r, \theta) = \nabla \times \mathbf{A}(r, \theta)$$

$$= \frac{\mu_0 m}{4\pi r^3} [2\hat{\mathbf{r}} \cos \theta + \hat{\boldsymbol{\theta}} \sin \theta]$$

Another approach

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \int \frac{d\ell'}{|\mathbf{x} - \mathbf{x}'|}$$

$$d\ell' = R d\phi' (-\hat{\mathbf{i}} \sin \phi' + \hat{\mathbf{j}} \cos \phi')$$

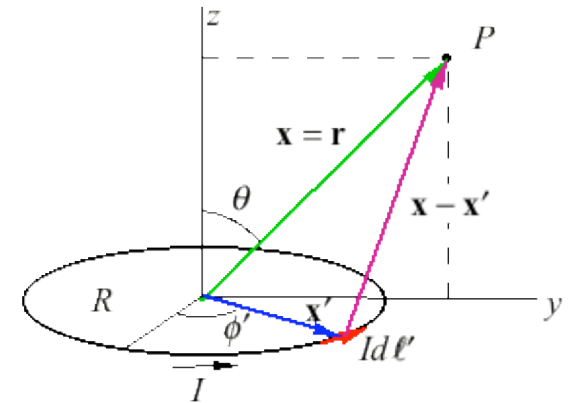
$$\mathbf{x} = \hat{\mathbf{j}} y + \hat{\mathbf{k}} z$$

$$\mathbf{x}' = \hat{\mathbf{i}} R \cos \phi' + \hat{\mathbf{j}} R \sin \phi'$$

$$\mathbf{x} - \mathbf{x}' = -\hat{\mathbf{i}} R \cos \phi' + \hat{\mathbf{j}} (y - R \sin \phi') + \hat{\mathbf{k}} z$$

$$|\mathbf{x} - \mathbf{x}'|^2 = R^2 \cos^2 \phi' + (R \sin^2 \phi' + y^2 - 2yR \sin \phi') + z^2$$

$$|\mathbf{x} - \mathbf{x}'| = \sqrt{R^2 + y^2 + z^2 - 2yR \sin \phi'}$$



$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 I R}{4\pi} \int_0^{2\pi} d\phi' \frac{(-\hat{\mathbf{i}} \sin \phi' + \hat{\mathbf{j}} \cos \phi')}{\sqrt{R^2 + y^2 + z^2 - 2yR \sin \phi'}}$$

$$= \frac{\mu_0 I R}{4\pi} \int_0^{2\pi} d\phi' \frac{(-\hat{\mathbf{i}} \sin \phi' + \hat{\mathbf{j}} \cos \phi')}{\sqrt{R^2 + r^2 - 2rR \sin \theta \sin \phi'}}$$

$A_y \hat{\mathbf{j}}$ @ $\phi' = 0, \pi$ have equal distance to P, average to zero

if point P is placed elsewhere $-\hat{\mathbf{i}}$ direction becomes $\hat{\phi}$

$$A_\phi = \frac{\mu_0 I R}{4\pi} \int_0^{2\pi} \frac{\sin \phi' d\phi'}{\sqrt{R^2 + r^2 - 2rR \sin \theta \sin \phi'}}$$

Now introduce Legendre Polynomials

Compare with expansion on slide 2

$$\cos \theta \rightarrow \sin \theta \sin \phi'$$

$$\frac{1}{\sqrt{r^2 + R^2 - 2rR \cos \theta}} = \frac{1}{r} + \frac{RP_1(\cos \theta)}{r^2} + \dots$$

$$A_\phi = \frac{\mu_0 IR}{4\pi r} \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^n \int_0^{2\pi} P_n(\sin \theta \sin \phi') \sin \phi' d\phi'$$

A = 0, for all even n, with **first non zero term, n = 1**

$$A_\phi \approx \frac{\mu_0 IR^2 \sin \theta}{4\pi r^2} \int_0^{2\pi} \sin^2 \phi' d\phi' = \frac{\mu_0 I \pi R^2 \sin \theta}{4\pi r^2}$$

$$A_\phi \approx \frac{\mu_0 m \sin \theta}{4\pi r^2}; \quad m = I \pi R^2$$

$$B_\theta = -\frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \approx \frac{\mu_0 m}{4\pi} \left(-\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\sin \theta}{r} \right) \right) = \frac{\mu_0 m}{4\pi r^3} \sin \theta$$

$$B_r = \frac{1}{r \sin \theta} \frac{\partial}{\partial r} (\sin \theta A_\phi) \approx \frac{\mu_0 m}{4\pi r^3 \sin \theta} \frac{\partial}{\partial \theta} (\sin^2 \theta) = \frac{\mu_0 m}{4\pi r^3} (2 \cos \theta)$$

$$\mathbf{B} \approx \frac{\mu_0 m}{4\pi r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}})$$

point dipole

$$\mathbf{B}_{\text{point dipole}} = \frac{\mu_0}{4\pi} \left[\frac{3\hat{\mathbf{r}}(\mathbf{m} \cdot \hat{\mathbf{r}}) - \mathbf{m}}{r^3} \right]; \quad \mathbf{m} = m \hat{\mathbf{k}}$$

General expressions for magnetic dipole

Dipole moment

$$\mathbf{m} = \frac{1}{2} \int \mathbf{x}' \times \mathbf{J}(\mathbf{x}') d^3 x'$$

Vector potential

$$\hat{\mathbf{m}} \times \hat{\mathbf{r}} = \int \mathbf{J}(\mathbf{x}') [\mathbf{x}' \cdot \hat{\mathbf{r}}] d^3 x'$$

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{\hat{\mathbf{m}} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi} \frac{\hat{\mathbf{m}} \times \mathbf{x}}{|\mathbf{x}|^3}$$

Magnetic field

$$\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \frac{3\hat{\mathbf{r}}(\mathbf{m} \cdot \hat{\mathbf{r}}) - \mathbf{m}}{r^3}$$

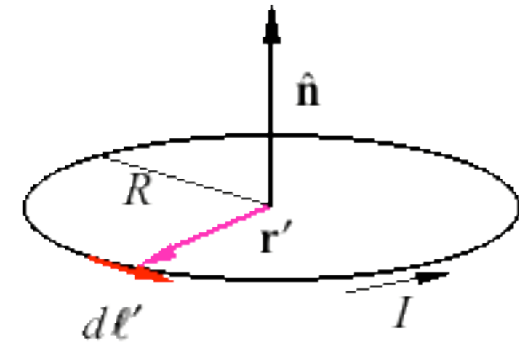
Torque on a magnetic dipole

$$\mathbf{N} = \mathbf{m} \times \mathbf{B}$$

Magnetic dipole potential energy

$$U = -\mathbf{m} \cdot \mathbf{B}$$

Circular loop



$$\mathbf{m} = \frac{IR}{2} \int \hat{\mathbf{r}} \times d\boldsymbol{\ell} = \frac{IR}{2} \int_0^{2\pi} R d\phi \hat{\mathbf{n}}$$

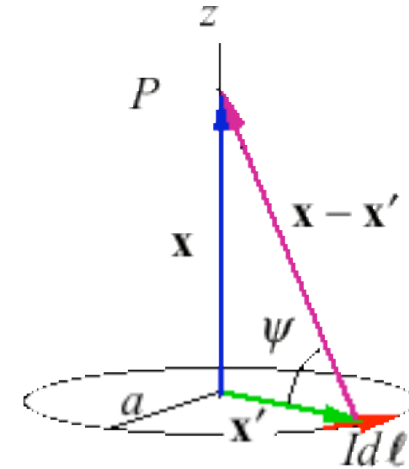
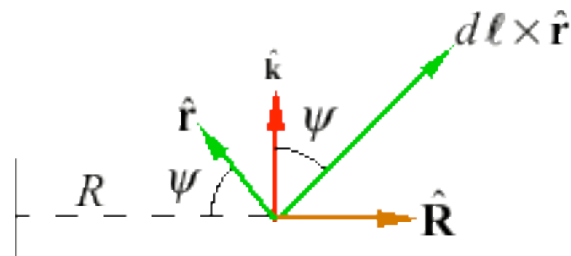
$$\mathbf{m} = I\pi R^2 \hat{\mathbf{n}} = IA\hat{\mathbf{n}}$$

HW8 hints

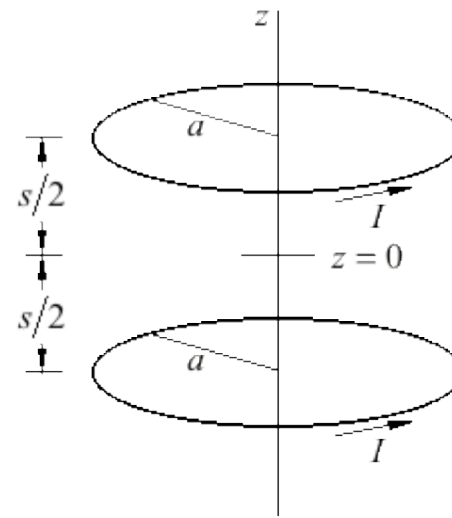
Problem 8.10

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}) \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi} \int \frac{d\boldsymbol{\ell} \times \hat{\mathbf{r}}}{r^2}$$

$$d\boldsymbol{\ell} = r d\phi \hat{\boldsymbol{\phi}}$$



$$\frac{d^2 B_z(0)}{dz^2} = 0$$

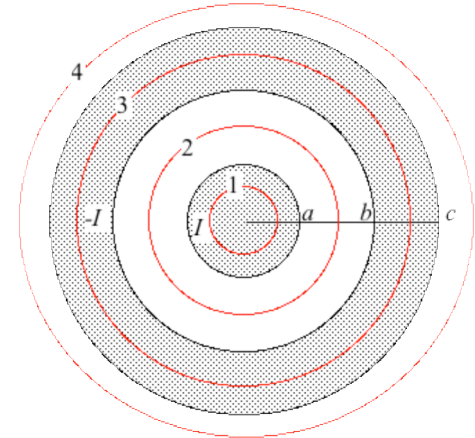


HW8 Hints

Problem 8.14 Constraints: $\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$

Problem 8.18

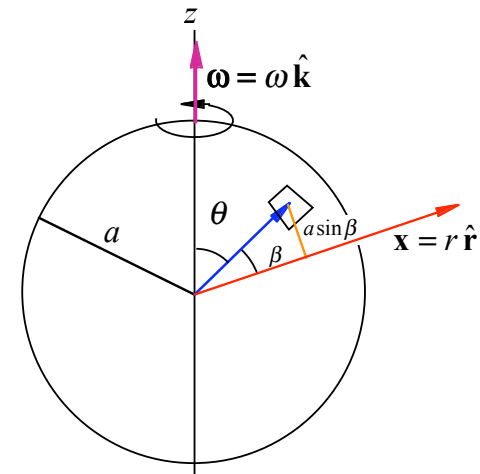
Ampere's Law: $\int \mathbf{B} \cdot d\ell = \mu_0 I_{encl} = \mu_0 \int \mathbf{J} \cdot d\mathbf{A}$



Problem 8.22 $\mathbf{A}_{in}$ & $\mathbf{A}_{out}$ given in text

$$\mathbf{B}_{in} = \nabla \times \mathbf{A}_{in} = \nabla \times \frac{\mu_0 \sigma a \omega}{3} (\hat{\mathbf{k}} \times r \hat{\mathbf{r}})$$

$$\mathbf{B}_{out} = \nabla \times \mathbf{A}_{out} = \nabla \times \left(\frac{\mu_0 \sigma a^4 \omega}{3} \right) \left(\hat{\mathbf{k}} \times \frac{\hat{\mathbf{r}}}{r^2} \right)$$



Problem 8.34 Use result of Problem 8.10