1. [5 pts] Imagine a peculiar cloud of cosmic dust whose mass distribution is spherically symmetric with a density (mass per unit volume) given by \( \rho(r) = k r^2 \) where \( r \) is the distance from the center and \( k \) is a constant. Find the gravitational force it produces on a “test mass” \( m \) at radius \( R \) which lies inside the dust cloud.
2. [5 pts] A particle of mass $m$ moves in a circular orbit of radius $R$ under the influence of a potential $U(r) = k r^3$ where $r = \sqrt{x^2 + y^2}$ and $k$ is a positive constant. Find its kinetic energy. (Hint: you can use the Effective Potential, or you can find the force and use the methods of first year physics – your choice.)
3. [5 pts] Find the gravitational potential energy of a point mass $m$ that is located a distance $a$ from a half-circle of wire mass $M$ as shown. The radius of the half-circle is $R$. You may leave your answer in the form of a well-defined integral—you do not need to evaluate the integral.
4. [5 pts] Derive the conditions under which \( \langle \hat{S} \rangle = 0 \), where \( S = \vec{p} \cdot \vec{r} \). (We used this in connection with the Virial Theorem.)
5. [5 pts] The equation for an elliptical orbit can be written in the form

\[ \frac{1}{r} = C_1 + C_2 \cos \theta. \]

Express this equation in terms of Cartesian \((x, y)\) coordinates and use your result to derive the condition on \(C_1\) and \(C_2\) for the orbit to be elliptical.
6. [5 pts] A satellite of small mass \( m \) is in a circular orbit of radius \( R \) about a planet of large mass \( M \). At a certain instant of time, a powerful rocket motor on the satellite is fired briefly, so as to increase the velocity without changing its direction. As a result, the satellite goes into an elliptical orbit as shown. Find the total energy (kinetic + potential) in the elliptical orbit in terms of \( G, m, M, \) and \( R \).