Homework 9

due Nov. 26 (after Thanksgiving)

- [CO 17.6 parts (a), (b)] [16.6 1st ed., parts (a), (b)] *Derive the deflection of the light ray passing a massive object.*

  Note that your answer will come out a factor of two smaller than Eq. 28.20 [26.15 1st ed.] because you are only asked to consider the term affecting the space-like part of the Schwarzschild metric. The time-like part is also affected (this is the answer to part c of this problem.

  Special offer: *I have placed copious hints for how to solve this problem on the course web site. You're in luck… I don't know how to set up a secure web site requiring credit card payments, so these hints are absolutely free of charge. But give the problem a try without the hints as a starter.*

- [CO 28.15] [26.16 1st ed.] *Pay attention to the hint given with the problem. And remember the following oldies but goodies:*

$$ax^2 + bx + c = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and the law of sines: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

- [CO 28.16] [26.17 1st ed.]

Dark Matter
so far

Mass/Luminosity

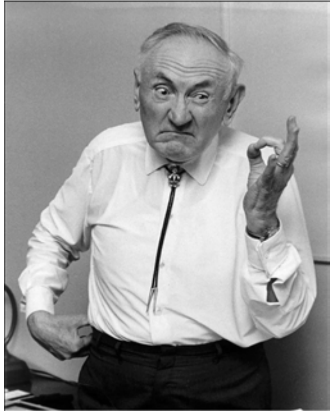
- Local stellar luminosity function: $M/L = 0.67$
- Our Galaxy, at larger scales:
  - Oort limit: $M/L \sim 2.7$
  - Slice through disk (Bahcall & Soniera) $\sim$ 5
  - Rotation curve > 30
  - Escape speed > 30
  - Pop II dynamics (glob. clusters, etc.) ~ 27
  - Magellanic stream > 80
  - Local Group timing 100
  - X-ray halo of M87 > 750
  - Groups of galaxies $200h$

Virial Theorem for Clusters

- Galaxy clusters – “fair samples” of the universe.

- Coma is closest relaxed cluster
- Original mass measurement was by Zwicky (1933).

$$M = \frac{5\sigma_r^2 R}{G} = 3\times10^{15} M_\odot$$



- Measure $n(r)$, $\sigma_v(r)$

  $n(r)$ = # of galaxies,

  $\sigma_v(r)$ = vel. Dispersion

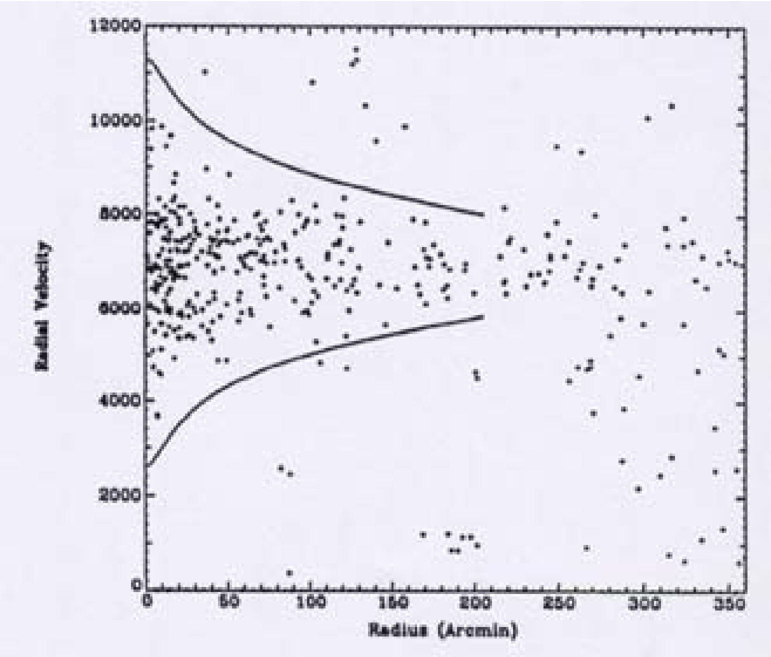
- Fit to models based on

  collisionless Boltzmann eq.

  ~ isothermal, non-spherical.

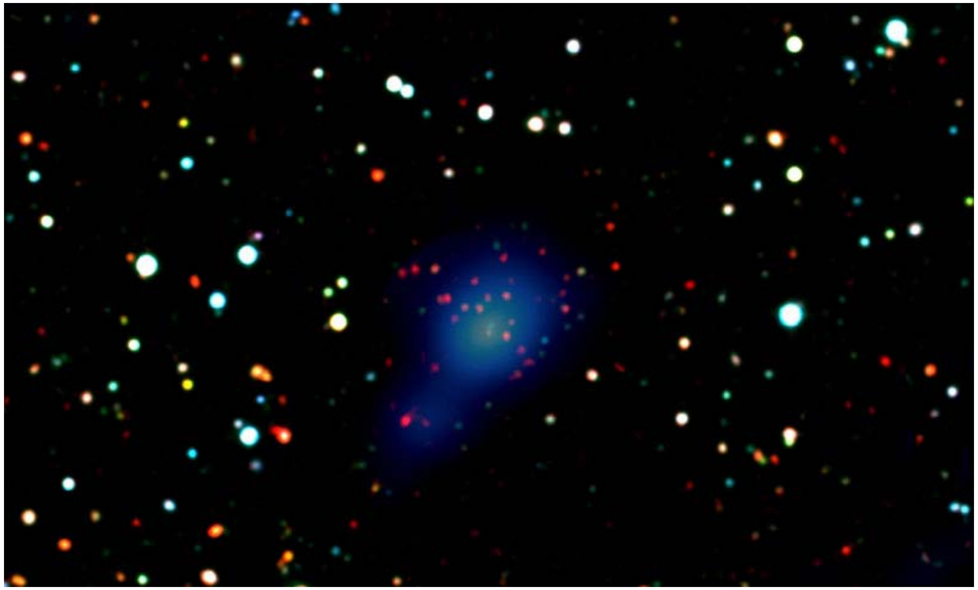
- Coma: $M = 2\text{x}10^{15}\,M_\odot$

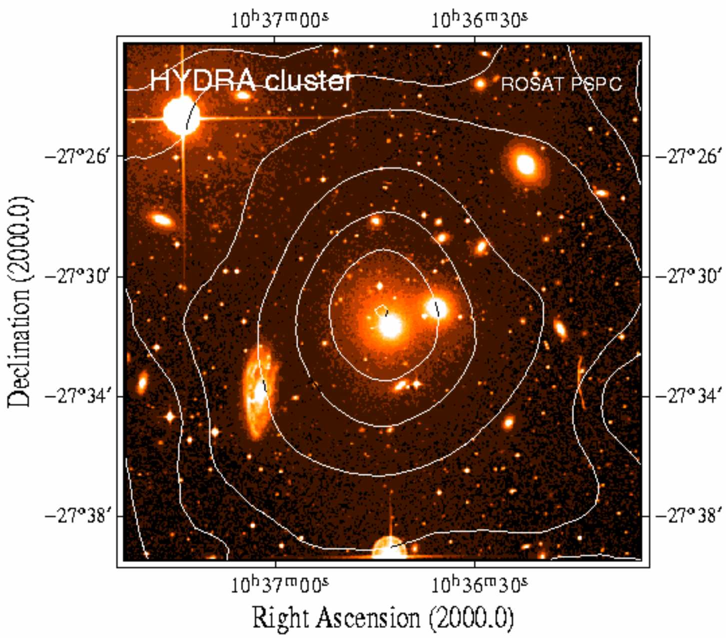
  $M/L = 360h$ (+0, -180h)
- Perseus: $M/L = 600h$

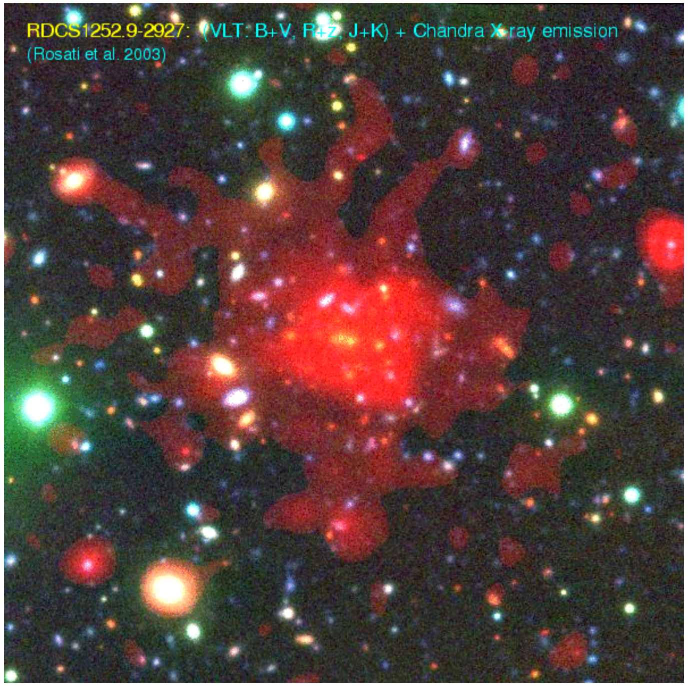


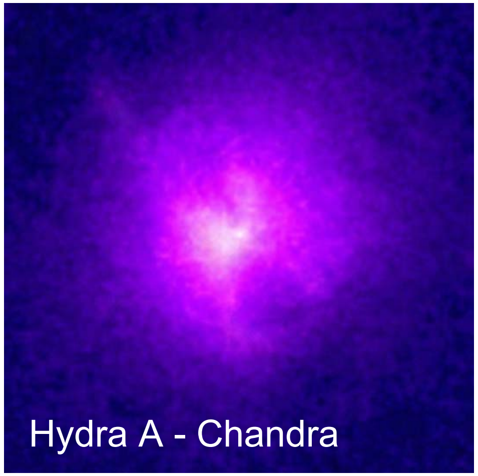
Determining membership

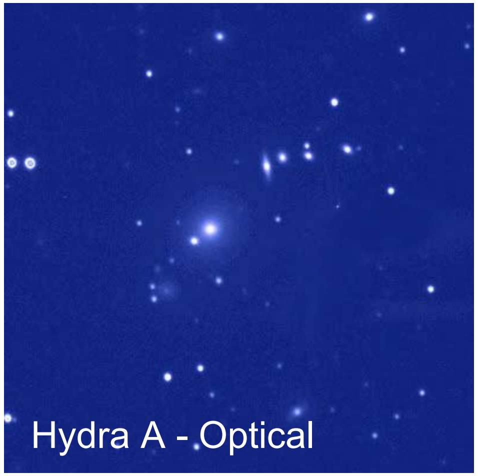
X-ray emitting gas in clusters







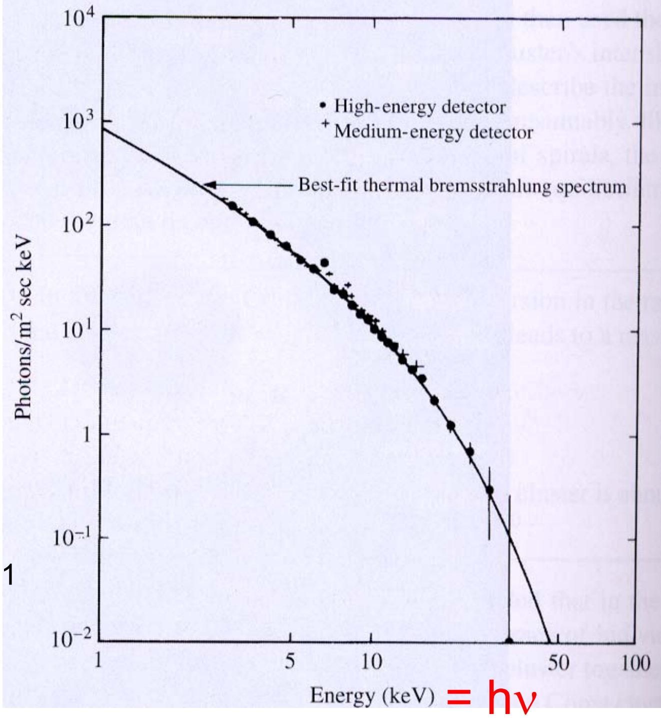




X-ray emitting gas in clusters

- Same method as used for x-rays from individual galaxies
  ==> $M/L \sim 180h$
- **gas is important mass component of cluster**
  - emission by thermal bremsstrahlung (free-free).
  - $L_X \sim 10^{43} - 10^{45}$ erg/s   (5×10^{44} erg/s for Coma)

[CO fig. 27.17]



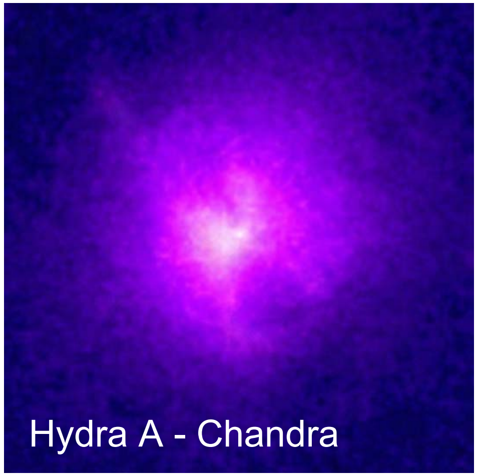
= hν

$$\ell_\nu d\nu = 5.44\times10^{-39}\overbrace{(4\pi n_e^2)T^{-1/2}}^{\text{amplitude}}\overbrace{e^{-h\nu/kT}}^{\text{freq. distr.}}d\nu \ \text{erg s}^{-1}\text{cm}^{-3}\text{Hz}^{-1}$$

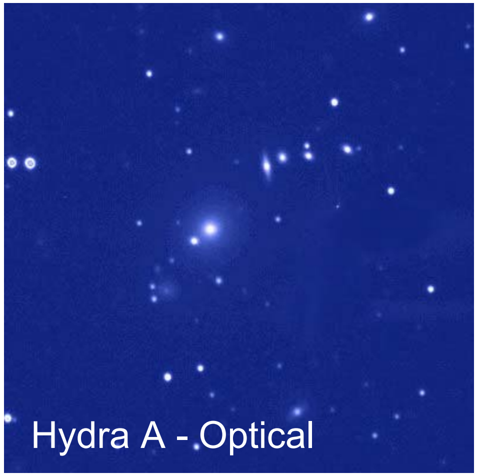
$$L_x = \frac{4}{3}\pi R^3 \int \ell_\nu d\nu = \frac{4}{3}\pi R^3 \times 1.42\times10^{-27} n_e^2 T^{1/2} \ \text{erg/sec}$$

measure L_x, R,T solve for n_e

- $M_{\text{gas}} = (4/3)\,\pi R^3 n_e m_H = 3\times10^{14}\ M_\odot$
- $M_{\text{stars}} = (M/L)_{Local}\, L_V = 2\times10^{13}\ M_\odot$



Hydra A - Chandra



Hydra A - Optical

Gravitational Lensing

- Foreground cluster distorts images of numerous background galaxies.
- Use to determine total mass of foreground cluster.
- Shows that 85% of mass is Dark Matter.

Gravitational Lensing

TWINKLE, TWINKLE LITTLE STAR
HOW I WONDER WHERE YOU ARE.
"1.75 SECONDS OF ARC FROM WHERE I SEEM TO BE
FOR $ds^2 \simeq (1 - 2GM/r^2)dt^2 - (1 + 2GM/r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2$."
Source unknown

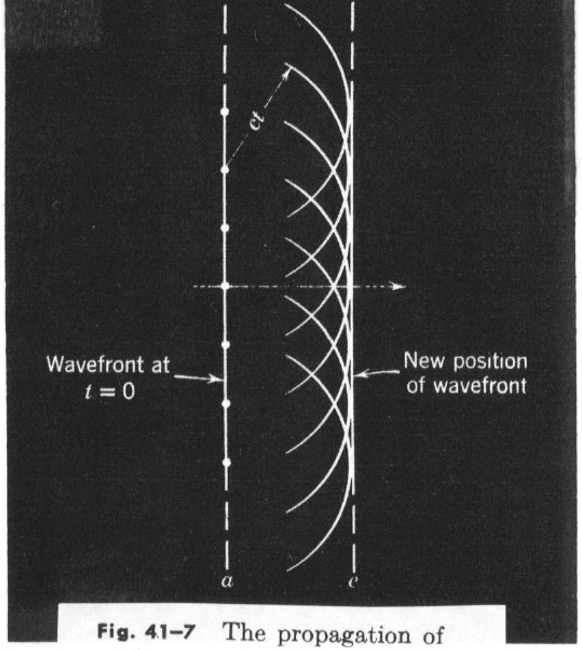


Fig. 41–7 The propagation of a plane wave in free space is described by the Huygens construction. Note that the ray (horizontal arrow) representing the wave is perpendicular to the wavefronts.

Metric for uniform distribution of mass: R-W metric

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\, d\theta)^2 + (\varpi \sin\theta\, d\phi)^2\right]$$

Metric for spacetime around point mass:

$$(ds)^2 = \left(c\,dt\sqrt{1-\frac{2GM}{rc^2}}\right)^2 - \left[\left(\frac{dr}{\sqrt{1-\frac{2GM}{rc^2}}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\, d\phi)^2\right]$$

= Schwarzschild metric

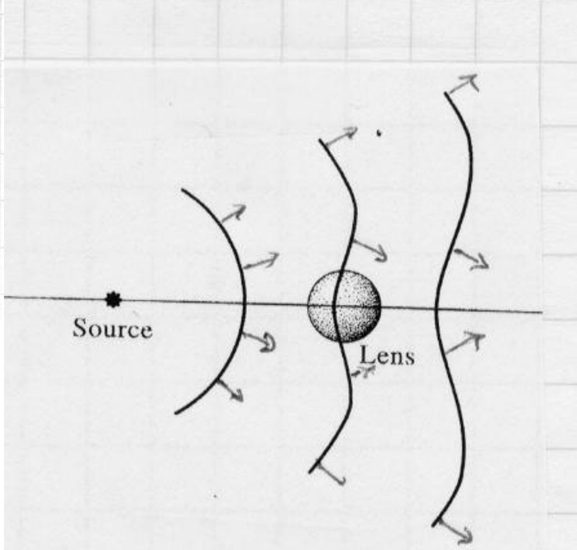
For light: $ds = 0$. if $d\theta = d\phi = 0$

$$\frac{dr}{dt} = c\left(1 - \frac{2GM}{rc^2}\right)$$

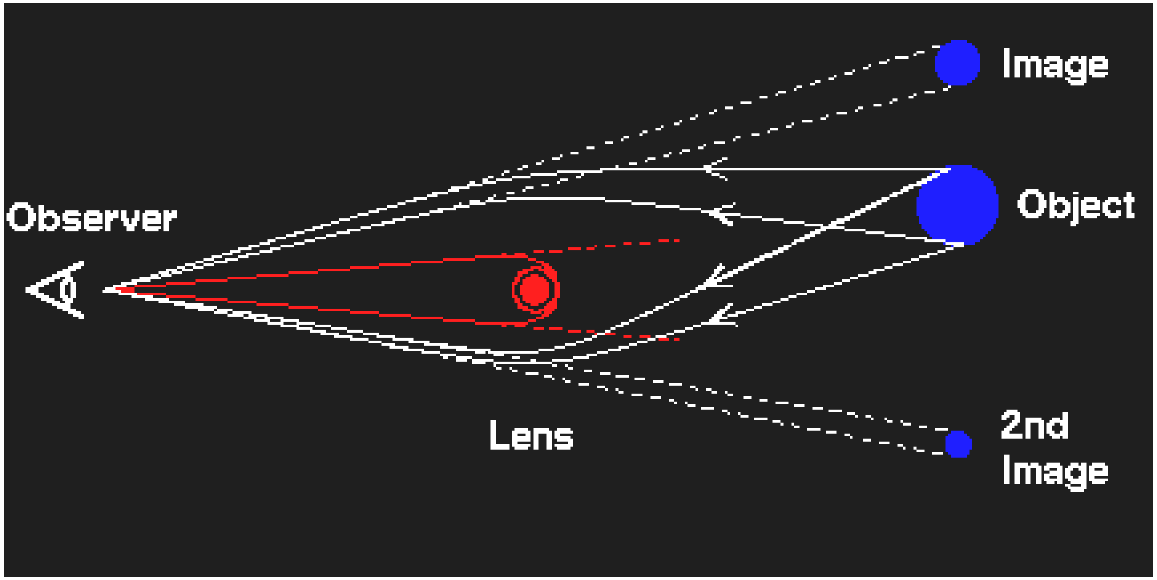
⇒ wavefront is retarded near a massive object

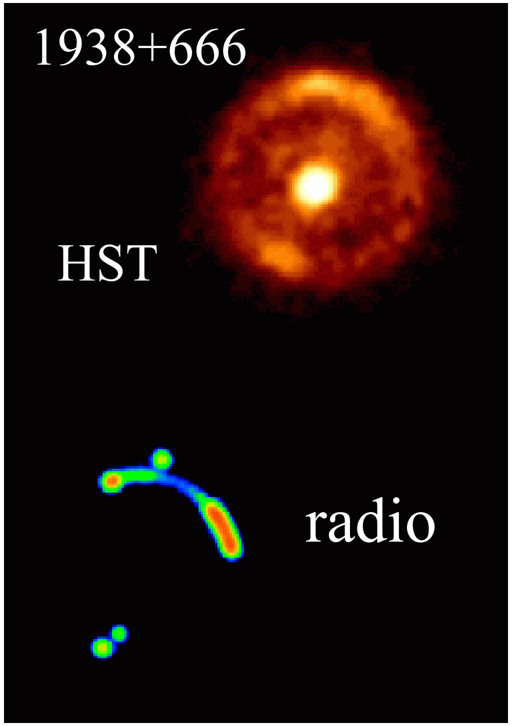
[CO 17.28]

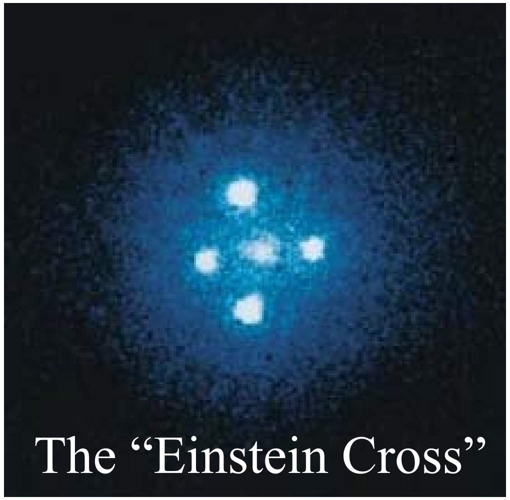
⇒ path of light is bent.



Gravitational Lenses



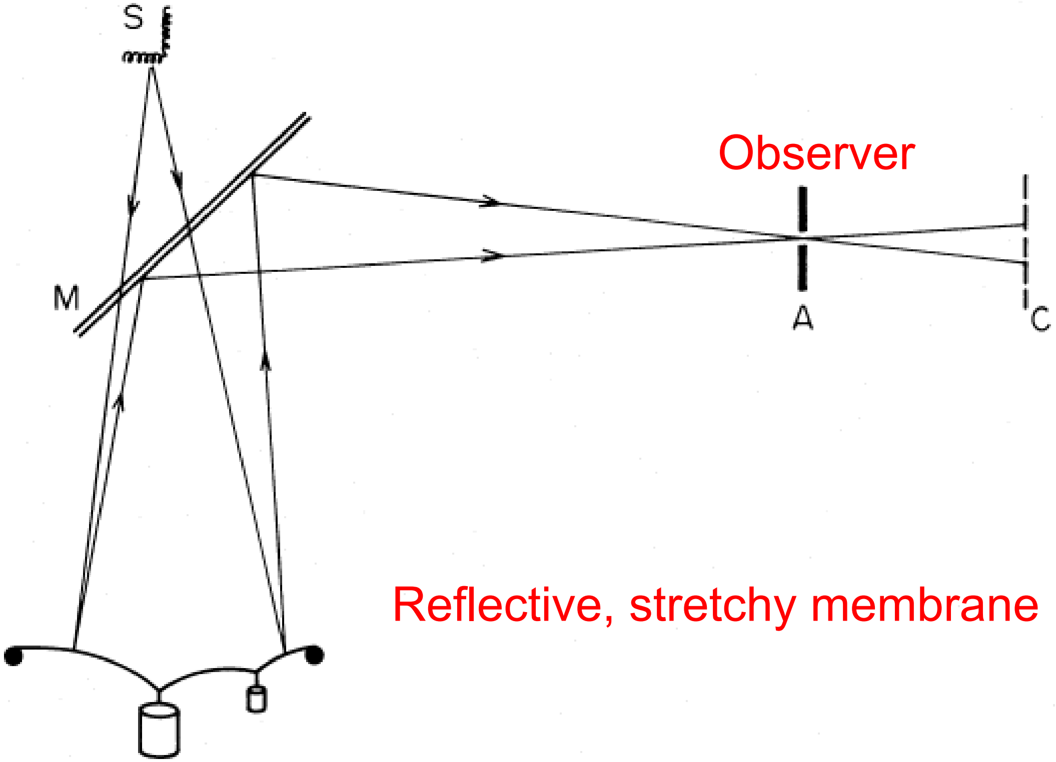



The “Einstein Cross”

Galaxy at center causes 4 images of same quasar.

Gravitational Lens Simulator

Blandford & Narayan 1986 ApJ, 310, 568



Source

Observer

Reflective, stretchy membrane

Lensing Mass

Gravitational Lensing by a Point Mass

[CO Sect. 28.4]

Angle of deflection of photon: $\phi = \frac{4GM}{r_0 c^2}$ (28.20)

$\Rightarrow \theta^2 - \beta\theta - \frac{4GM}{c^2}\left(\frac{d_S - d_L}{d_S d_L}\right) = 0$ (from trig) (28.21)

Quadratic eq. in θ $\Rightarrow$ 2 solutions θ_1, θ_2

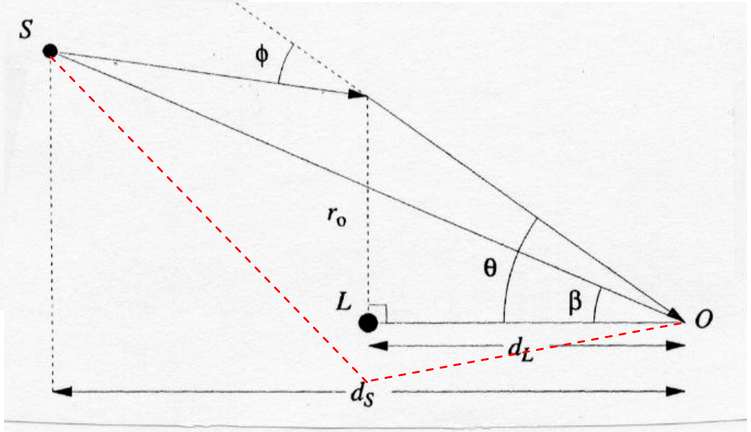
$$\beta = \theta_1 + \theta_2$$

$$M = -\frac{\theta_1 \theta_2 c^2}{4G}\left(\frac{d_S d_L}{d_S - d_L}\right)$$

If lens is exactly on line of sight to source: $\beta = 0$

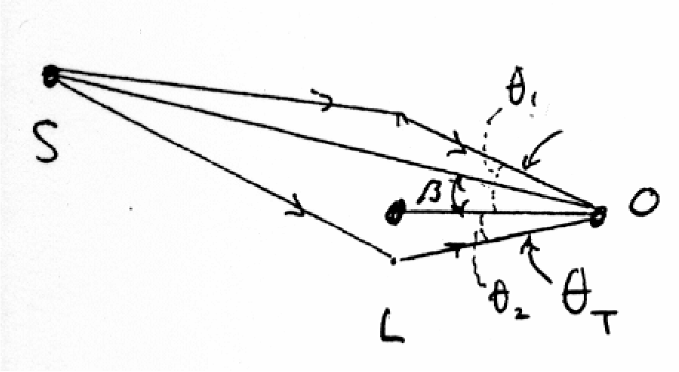
image is Einstein Ring

with $\theta_E = \sqrt{\frac{4GM}{c^2}\left(\frac{d_S - d_L}{d_S d_L}\right)}$ (28.24)



Effect of Lensing on Flux

See Refsdal (1964) MNRAS 128, 295



Use [CO] notation, but also define

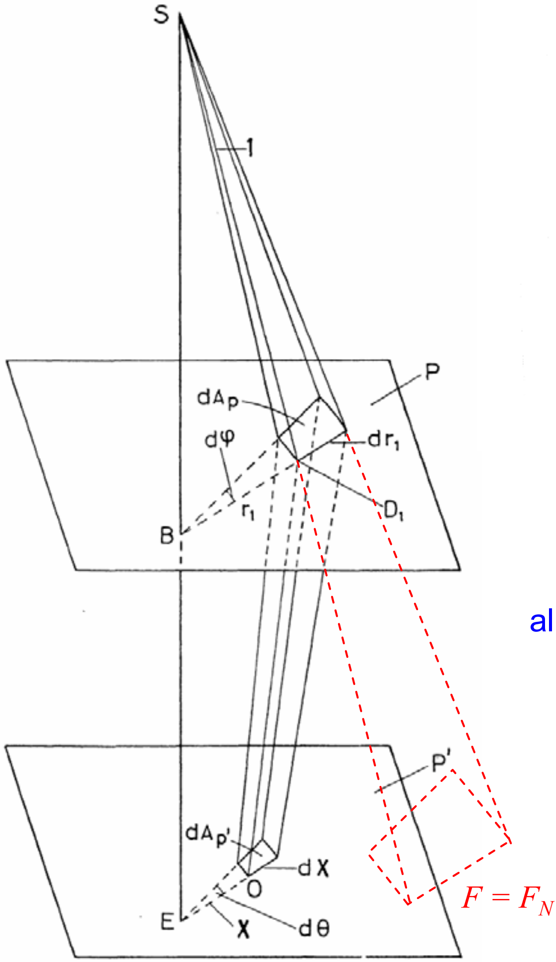
$\theta_T = |\theta_1| + |\theta_2|$ = total separation between images

$$\theta_T = \sqrt{\theta_E^2 + \beta^2}$$

$$F_1 = \frac{1}{4}\left(2 + \frac{\theta_T}{\beta} + \frac{\beta}{\theta_T}\right) F_N$$

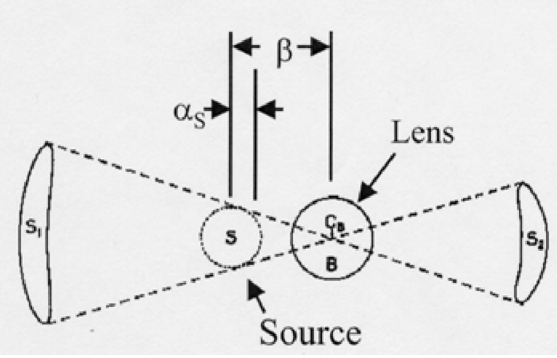
$$F_2 = \frac{1}{4}\left(-2 + \frac{\theta_T}{\beta} + \frac{\beta}{\theta_T}\right) F_N$$

$$F_{TOTAL} = F_1 + F_2 = \frac{1}{2}\left[\frac{\theta_T}{\beta} + \frac{\beta}{\theta_T}\right] F_N$$



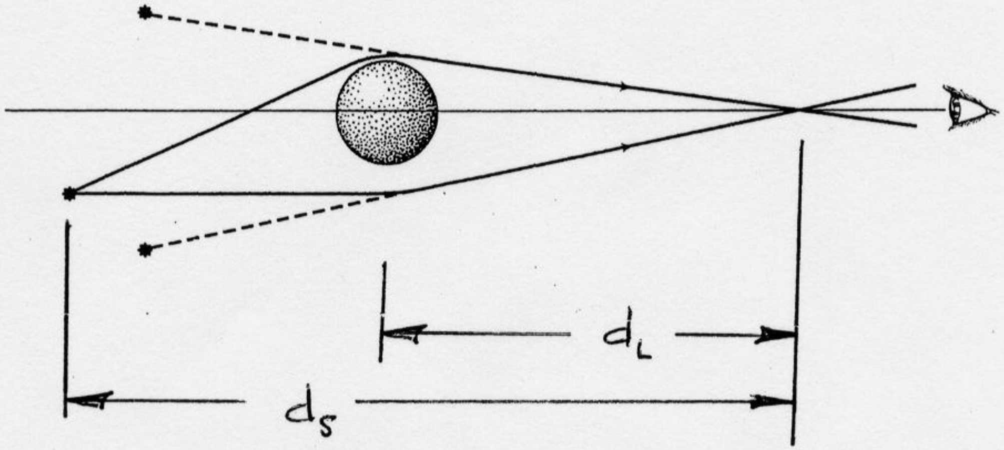
Lensing of Extended Sources

- Image has same surface brightness as unlensed image, but more area.
- Ring if $\beta < \alpha_S$
- Arcs if $\beta > \alpha_S$
- Max amplification when $\beta = 0$ $\sim \theta_E/\alpha_S$



	β/θ_E	F_1/F_N	F_2/F_N	F_T/F_N
Not aligned	10	1·0000063	0·0000063	1·000013
	5	1·0001	0·0001	1·0002
	3	1·0014	0·0014	1·0028
	1·5	1·0084	0·0084	1·017
$\beta = \theta_E$	1	1·030	0·030	1·06
	0·6	1·116	0·116	1·23
	0·4	1·27	0·27	1·54
	0·3	1·44	0·44	1·88
	0·2	1·83	0·83	2·66
	0·15	2·23	1·23	3·46
	0·1	3·04	2·04	5·08
Close alignment	0·05	5·52	4·52	10·0
	0·01	25·5	24·5	50·0

Point mass forms two images (or ring)



- For sun: rays intersect at $d_L \sim 50$ ly

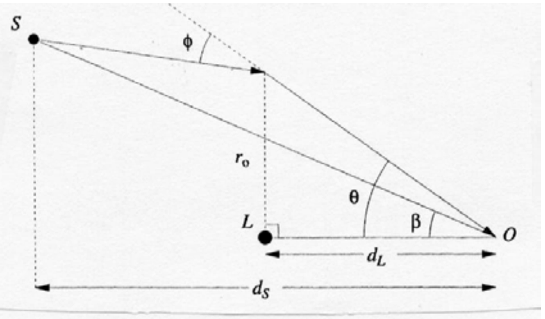
- For $d_S \gg d_L$

$$\theta_E = \sqrt{\frac{4GM}{c^2}\left(\frac{d_S - d_L}{d_S d_L}\right)} = \sqrt{\frac{4GM}{c^2 d_L}}$$

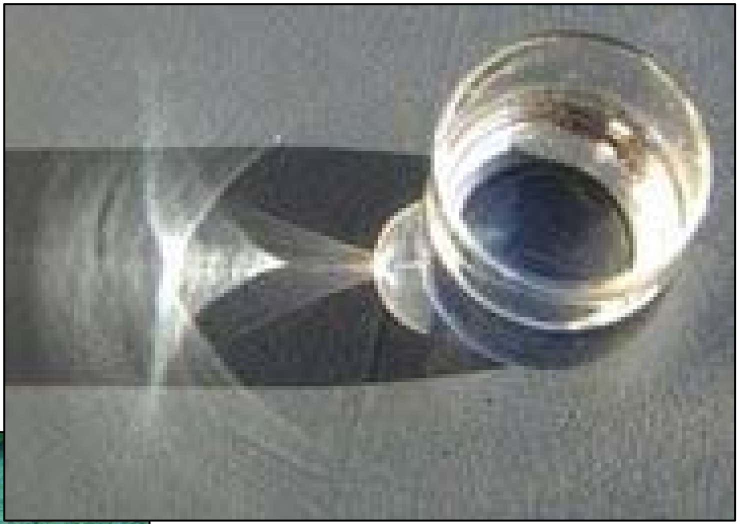
- For stars in Milky Way:
  - $M = 1\ M_{sun}$, $d_L = 10^4$ ly $\Longrightarrow$ $\theta_E \sim 2 \times 10^{-3}$ arcsec

- For external galaxies
  - $M = 10^{11}\ M_{sun}$, $d_L = 10^{10}$ ly $\Longrightarrow$ $\theta_E \sim 1$ arcsec

- Need $\beta < \theta_E$ to see multiple images (strong lensing)



Caustics & Catastrophes



Lensing by a Transparent Mass Distribution

WAVEFRONT RETARDED BY GRAVITATIONAL FIELD:

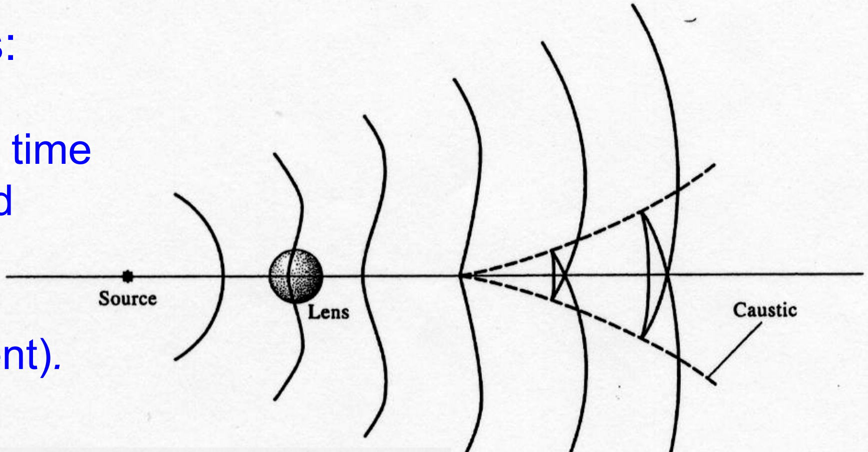
POINT MASS:
$$(ds)^2 = \left(c\,dt\sqrt{1-2GM/rc^2}\right)^2 - \left(\frac{dr}{\sqrt{1-2GM/rc^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$

for light:
$$\frac{dr}{dt} = c\left(1-2GM/rc^2\right)$$

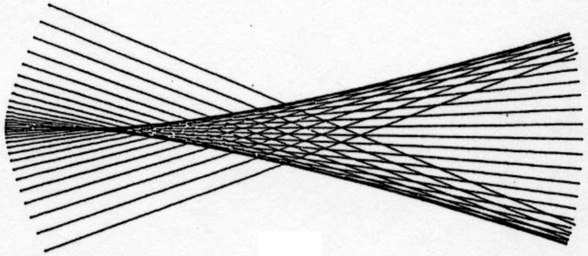
TRANSPARENT MASS DISTRIBUTION:
$$(ds)^2 = (c^2+2\Phi)\,dt^2 - \left(1-\frac{2\Phi}{c^2}\right)(dx^2+dy^2+dz^2)$$

Caustic Surfaces:

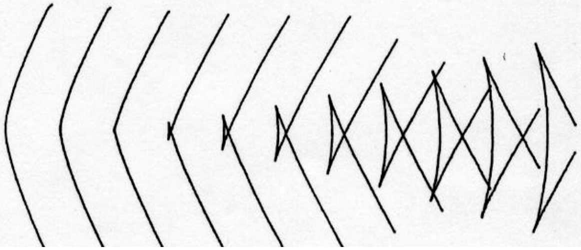
Number of images changes by 2 each time a caustic is crossed
➔always an odd number
(*if* lens is transparent).

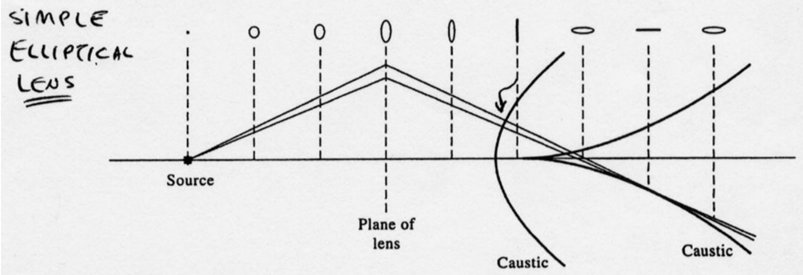


Rays:



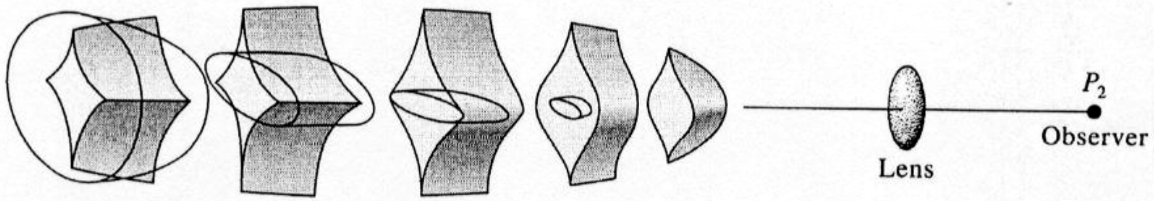
Wave fronts:





Conjugate caustic surfaces

– elementary catastrophes, etc.



- fig 4.22, Blandford figs 6, 7.

Fig. 4.22 Conjugate caustic surfaces for a lens consisting of an ellipsoidal mass. (After Blandford and Narayan, 1986.)

BLANDFORD & NARAYAN

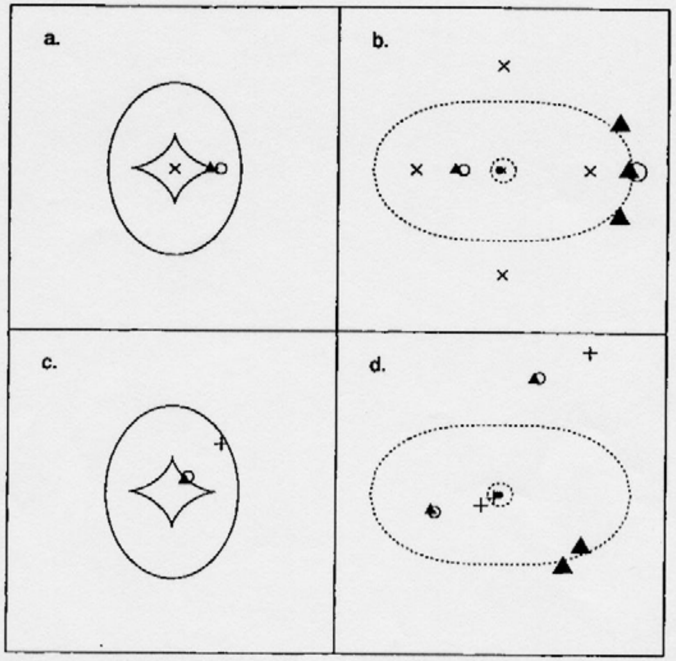
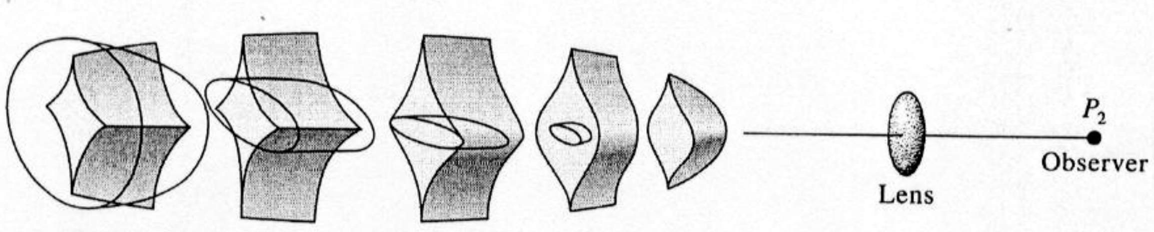


Figure 6 Multiple imaging of point sources at fixed redshift by a generic "elliptical lens." The solid lines in the left panels are *caustics* that separate regions in the source plane corresponding to different image multiplicities (1, 3, and 5 as indicated). The inner caustic, sometimes referred to as the *tangential caustic*, has four cusps connected by fold lines. The outer *radial caustic* is a pure fold. The outer dashed lines in the right panels are tangential critical curves and the inner ones are radial critical curves. The symbols show representative source positions and the corresponding image locations. When the source is close to a caustic, some of the images are strongly magnified, indicated by large symbols in the image panels. One of the multiple images usually occurs near the center of the lens and is strongly demagnified if the core radius of the lens is small. Among the "secure" multiple quasars, Q1413+117 and Q2237+031 correspond to the source position × and Q0142−100 to ○ in the upper panels. 0414+053 and Q1115+080 correspond to the triangle and Q0957+561 is midway between ○ and + in the lower panels. The weak central image has not been seen in any of the observed cases.

Conjugate caustic surfaces

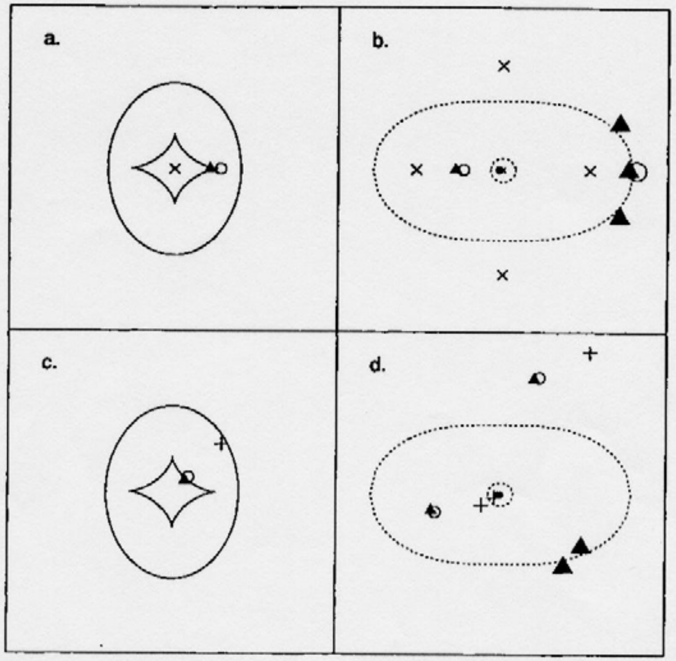
- elementary catastrophes, etc.



- fig 4.22, Blandford figs 6, 7.

Fig. 4.22 Conjug
(After Blandford a

BLANDFORD & NARAYAN



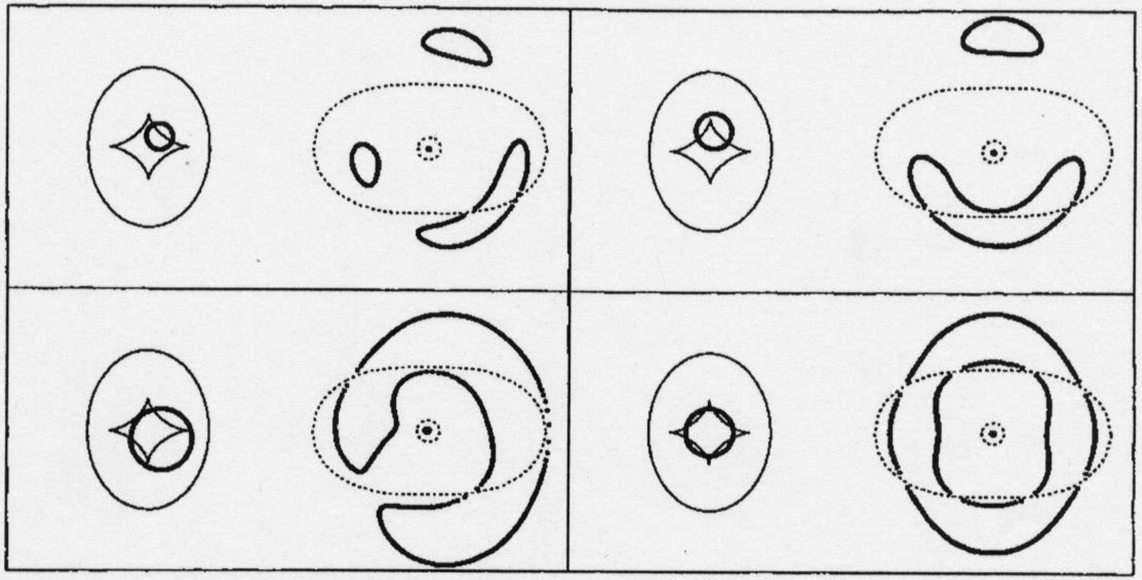


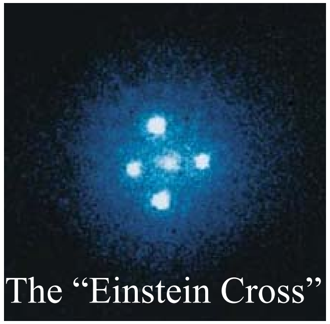
Figure 7 Representative arc and ring images of resolved sources produced by an elliptical lens. In each set, the source planes are on the left and the corresponding images are on the right. The long luminous arcs in Abell 370, Cl 2244−02, Abell 963 and other clusters are similar to the case displayed at top right. (The counter-image shown here will not be present for certain choices of the lens parameters; see Narayan & Grossman 1989, Narayan & Wallington 1992a.) The radio rings correspond to the case shown at bottom right, and the incomplete ring in MG1131+0456 at 15 GHz is similar to the example at bottom left.

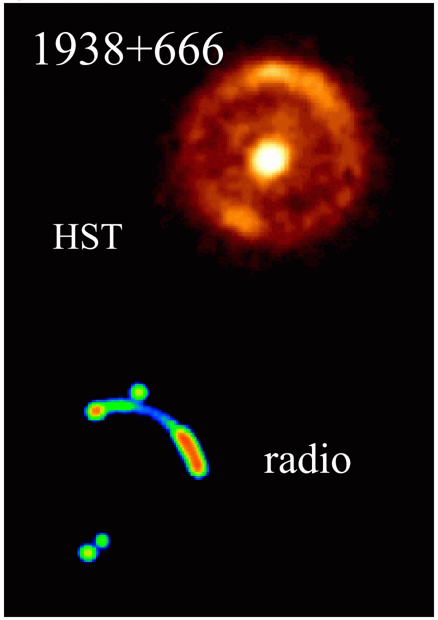
Observations of lensed objects

- Extended background source (e.g. a galaxy)

  ==> arcs or rings

- Weak lensing: $\theta >> \theta_E$
  - images slightly extended
  - currently being exploited to look for cluster halos, dark galaxies, etc.
- Strong lensing: $\theta < \theta_E$
  - multiple images formed
  - weak central image usually not seen


The "Einstein Cross"



Weak (and not-so-weak) Lensing
Abell 2218

- Foreground cluster distorts images of numerous background galaxies.
- Use to determine total mass of foreground cluster.
- Shows that 85% of mass is Dark Matter.

Using caustics to search for high-redshift background galaxies

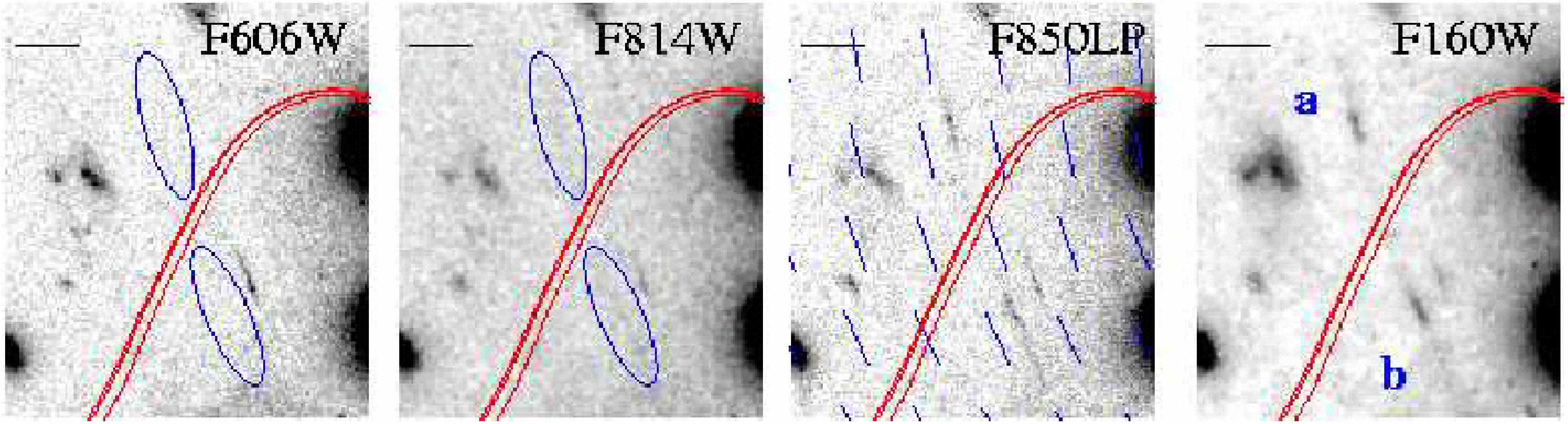
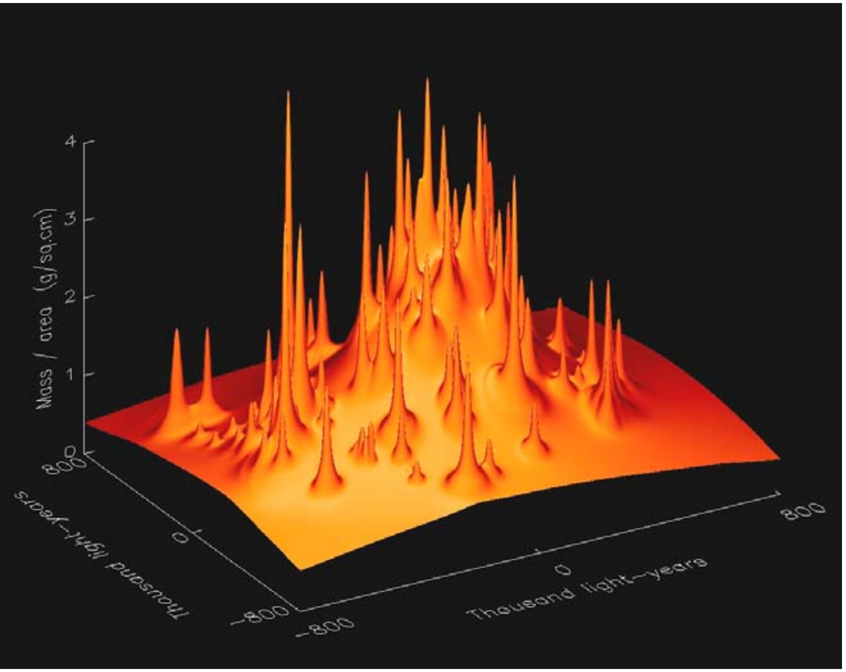
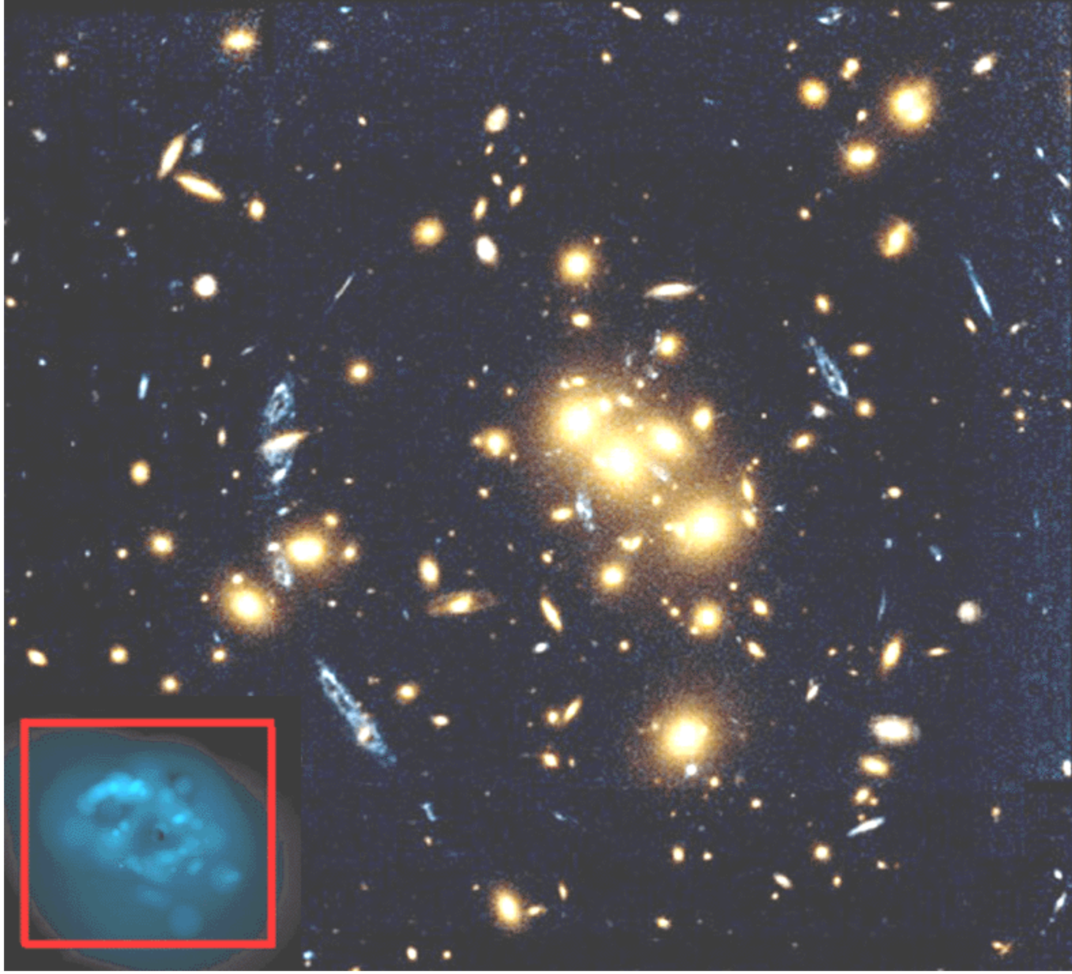


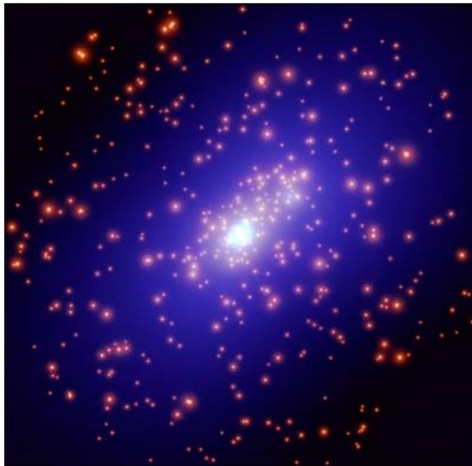
Figure 1: *WFPC2*-F606W, *WFPC2*-F814W, *ACS*-F850LP and NICMOS-F160W images of Abell 2218 of the new faint pair in the lensing cluster Abell 2218 (z=0.175). The signals redward of the *WFPC2*-F814W observation suggests a marked break occurs in the continuum signal at around 9600Å . Red lines correspond to the predicted location of the critical lines at z_s=5,6.5 and 7 (from bottom to top, the latter two being almost coincident). The scale bar at the top left of each image represents $2''$. The predicted shear direction (thin blue lines) closely matches the orientation of the lensed images.

The Remarkable Case of CL0024+1654

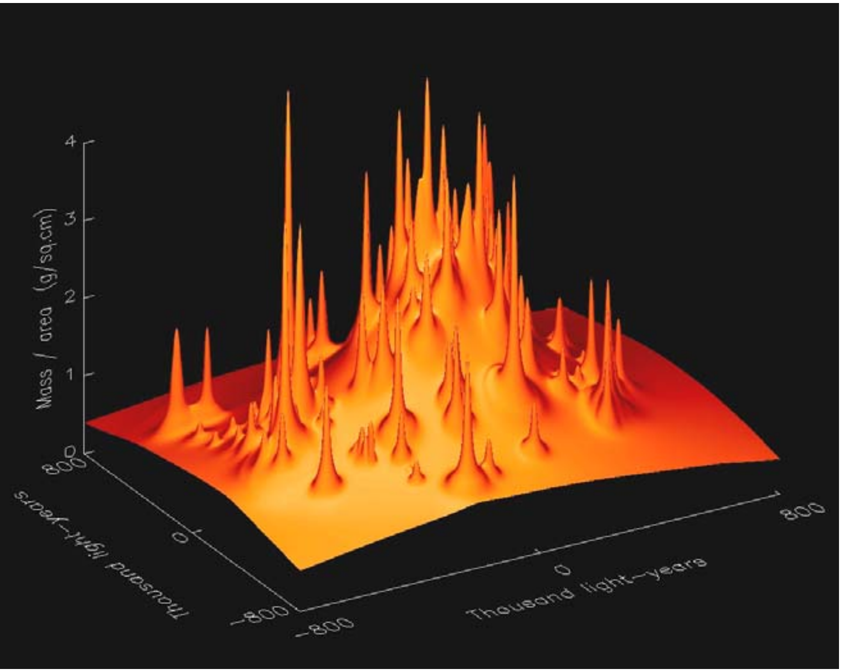


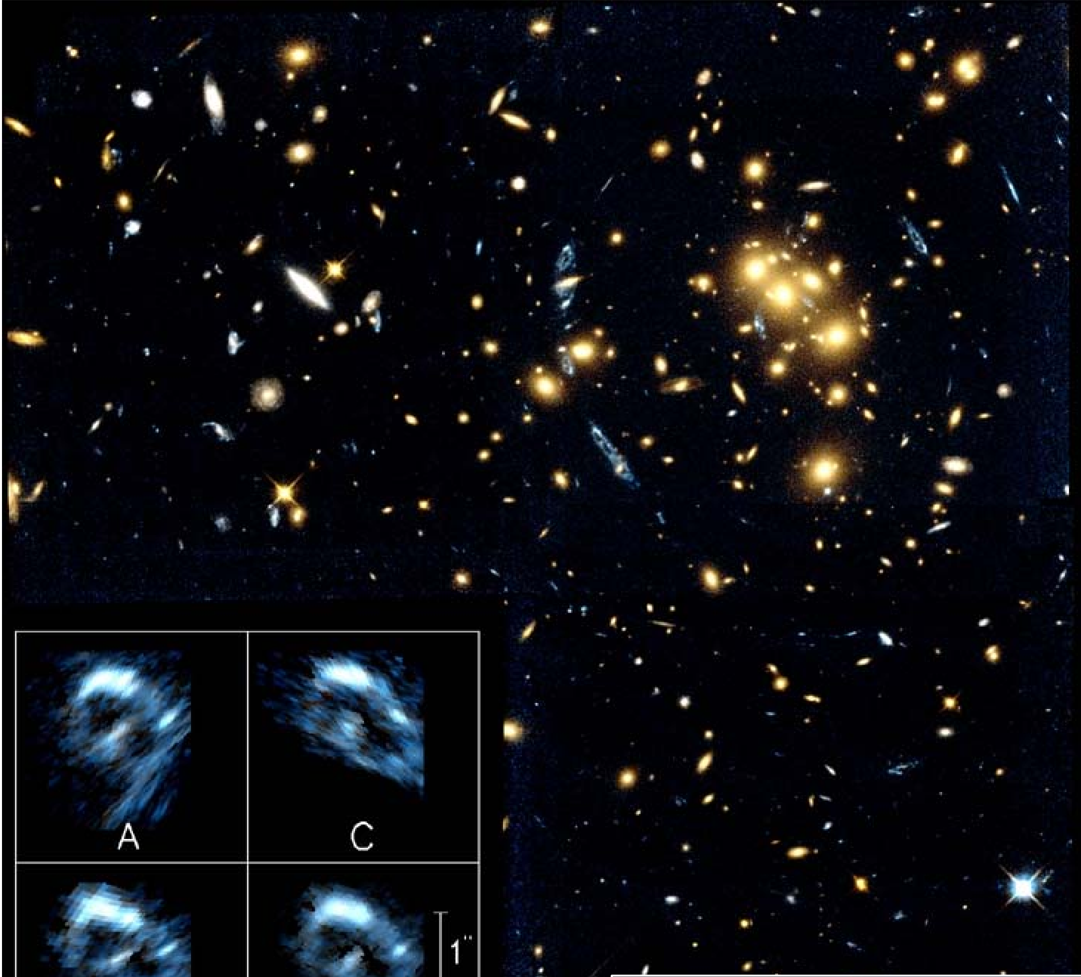


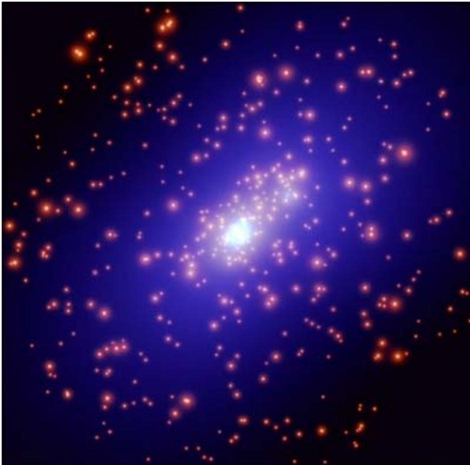
- Single distant blue galaxy
  - $z \sim 1.2 - 1.8$
- Lensed by foreground cluster
  - $z = 0.39$
- 8 different grav. images of blue galaxy.
- Allows detailed analysis of mass distribution in cluster.
- 83% of mass is non-luminous Dark Matter.
- $M/L = 270h$ (390h after allowing for stellar evolution to $z = 0$)



The Remarkable Case of CL0024+1654







- Single distant blue galaxy
  - $z \sim 1.2 - 1.8$
- Lensed by foreground cluster
  - $z = 0.39$
- 8 different grav. images of blue galaxy.
- Allows detailed analysis of mass distribution in cluster.
- 83% of mass is non-luminous Dark Matter.
- $M/L = 270h$ ($390h$ after allowing for stellar evolution to $z = 0$)

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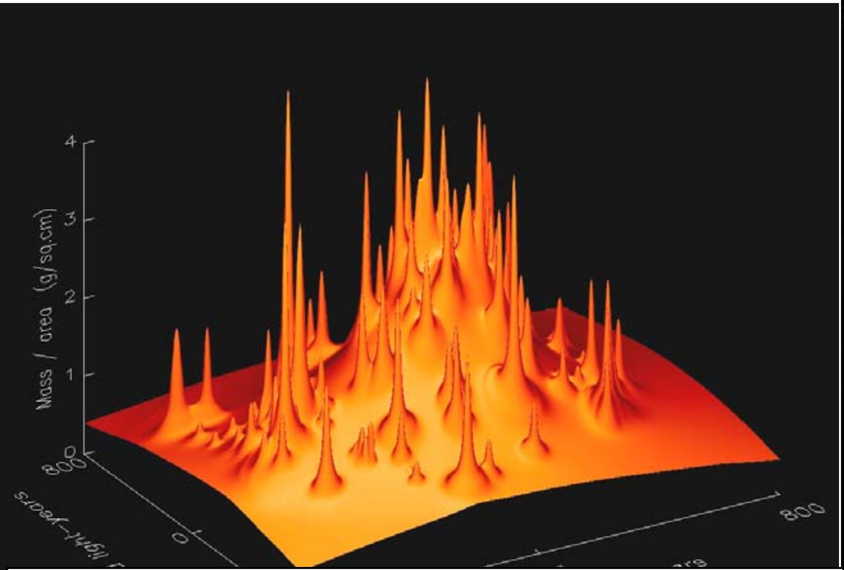
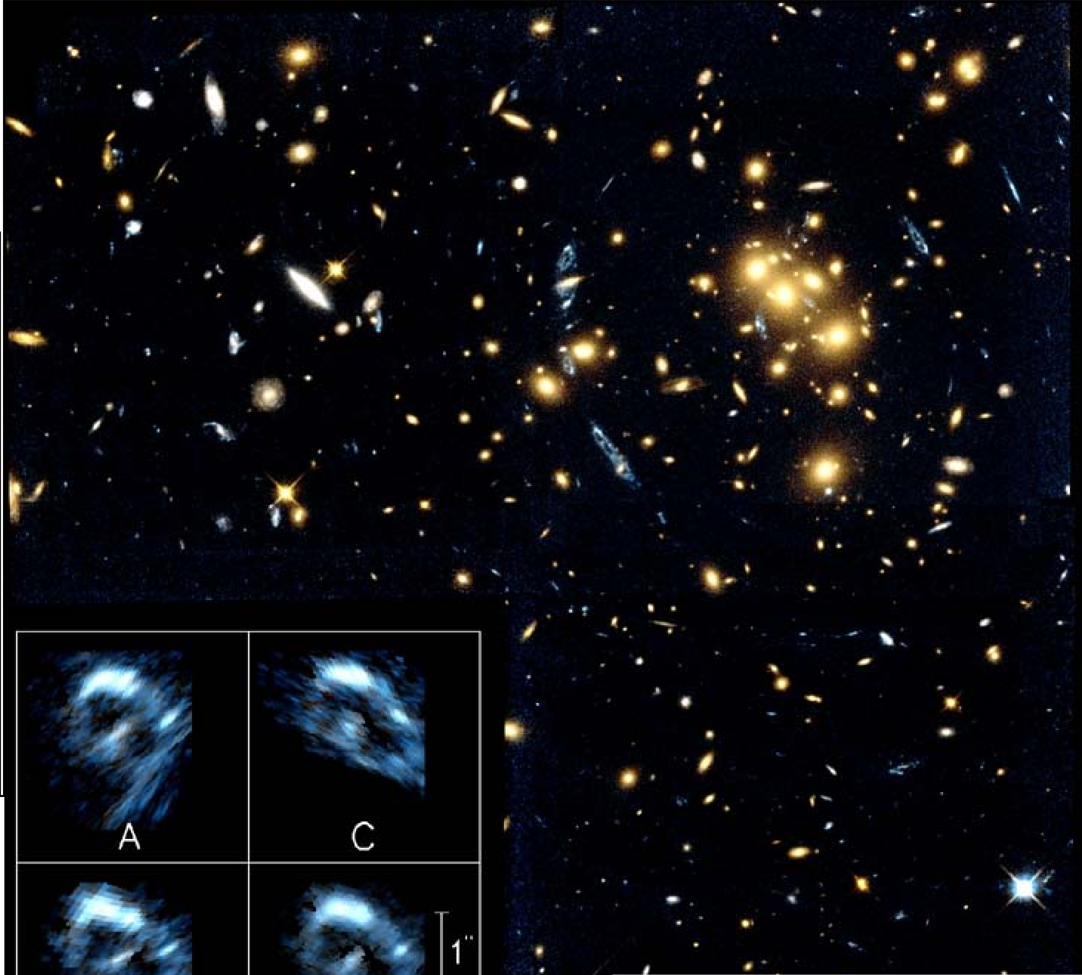
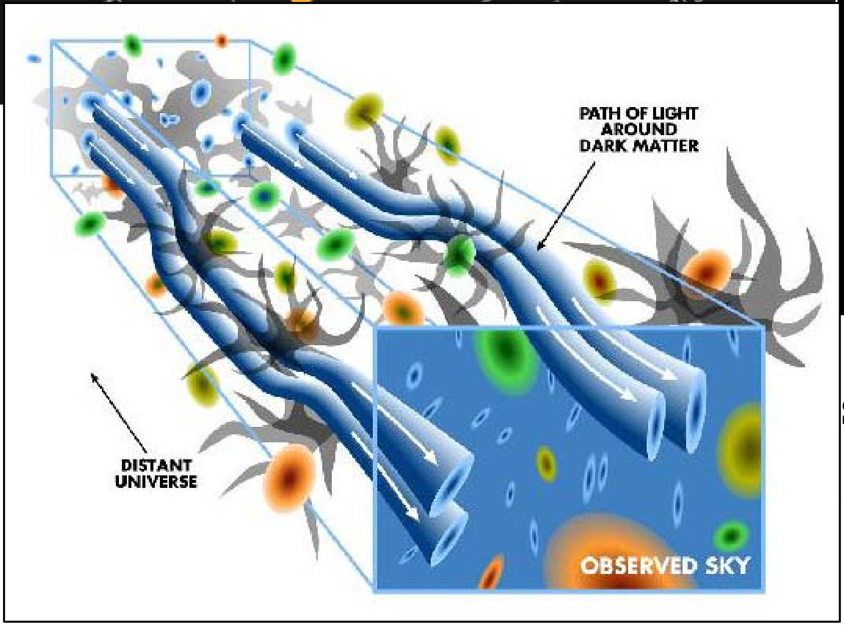
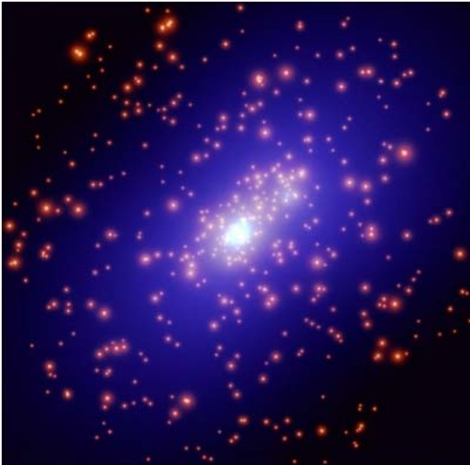






stribution in cluster.

Matter.



Dark Matter

Galaxy clusters, typical values

- 85% Dark Matter
- 15% Normal Matter
  - 14% hot intergalactic gas
  - ~1% stars in galaxies

Average for universe:
85% dark matter
15% normal matter

Mass/Luminosity

- Local stellar luminosity function: $M/L = 0.67$
- Our Galaxy, at larger scales:
  - Oort limit: $M/L \sim 2.7$
  - Slice through disk (Bahcall & Soniera) ~ 5
  - Rotation curve > 30
  - Escape speed > 30
  - Pop II dynamics (glob. clusters, etc.) ~ 27
  - Magellanic stream > 80
  - Local Group timing 100
  - X-ray halo of M87 > 750
  - Groups of galaxies $200h$
  - Clusters of galaxies $400h$