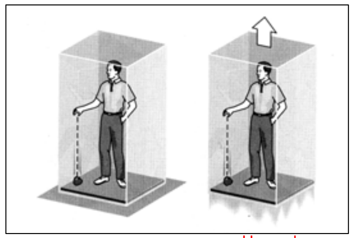
General Relativity (sort of)

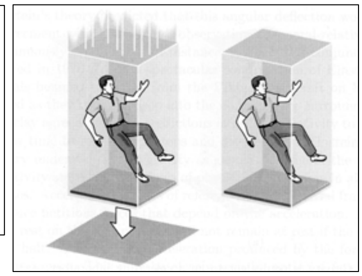
Equivalence Principle:

- Can't tell difference between gravity & acceleration
- …or between freefall & no gravity.
- So *any* experiment should give same answer in either case.



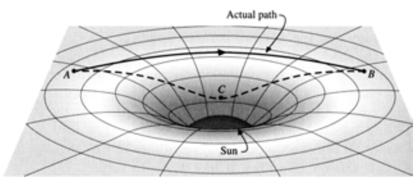
Gravity

Upwards acceleration, no gravity.

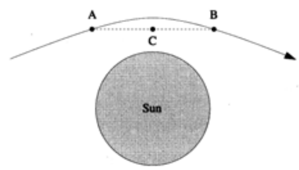


Falling due to gravity

No gravity



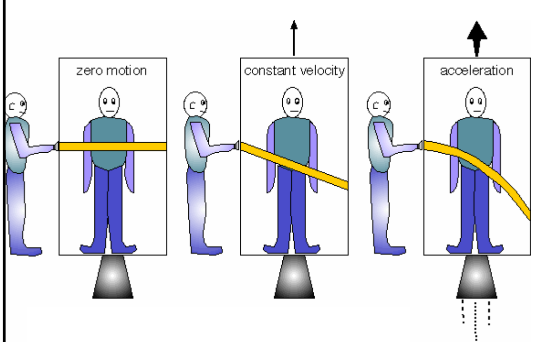
Gravity = Curved space

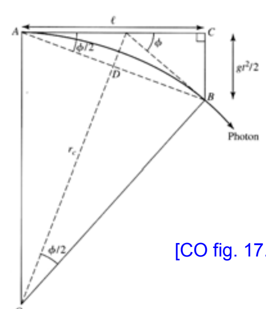


Objects follow shortest distance through curved space(-time).

The equivalence principle (plus a vigorous waving of one's hands) shows…

- Curved path of light in gravitational field





[CO fig. 17.10]

$$\overline{BC}/\overline{AC} = \overline{BD}/\overline{OD}$$

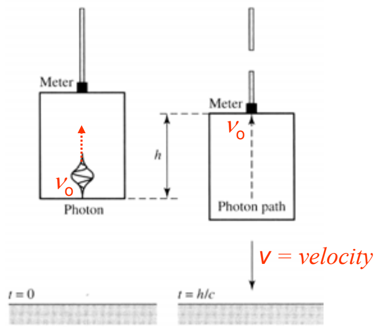
$$\left(\frac{1}{2}gt^2\right)\Big/\ell = \left[\frac{\ell}{2\cos(\phi/2)}\right]\Big/\overline{OD}$$

$$r_c = \frac{c^2}{g} = 9.17 \times 10^{15}\ \mathrm{m}$$

$$\phi = \frac{\ell}{r_c} = 1.09 \times 10^{-15}\ \mathrm{rad}$$

The equivalence principle (plus a vigorous waving of one's hands) shows…

- Gravitational Redshift



Equiv. Principle ➔ photon frequency unchanged in free-falling lab.

Why doesn't the frequency meter see a *blueshift*?

$$\frac{\Delta\nu}{\nu_0} = \frac{v}{c} = \frac{gh}{c^2}$$

There must be a counteracting *Gravitational Redshift*:

This redshift is seen by the meter which is *not* free-falling. ⇨

$$\frac{\Delta\nu}{\nu_0} = -\frac{v}{c} = -\frac{gh}{c^2}$$

Integrate the effect out to infinite distance:

$$\int_{\nu_0}^{\nu_\infty} \frac{d\nu}{\nu} \simeq -\int_{r_0}^{\infty} \frac{GM}{r^2c^2}\,dr$$

$$\frac{\nu_\infty}{\nu_0} \simeq 1 - \frac{GM}{r_0c^2}$$

The exact result:

$$\frac{\nu_\infty}{\nu_0} = \left(1 - \frac{2GM}{r_0c^2}\right)^{1/2}$$

- Gravitational time dilation

$$\frac{\Delta t_0}{\Delta t_\infty} = \frac{\nu_\infty}{\nu_0} = \left(1 - \frac{2GM}{r_0c^2}\right)^{1/2}$$

Special Relativity
The Lorentz Transformation

CO, pg. 90

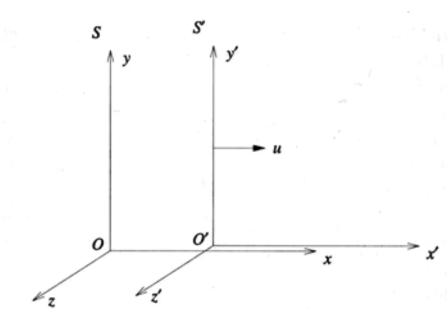


Figure 4.2 Inertial reference frames S and S'.

$$x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}} \quad (4.16)$$

$$y' = y \quad (4.17)$$

$$z' = z \quad (4.18)$$

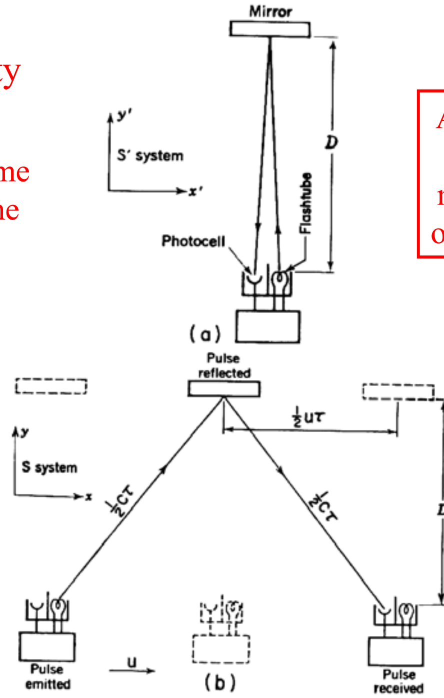
$$t' = \frac{t - ux/c^2}{\sqrt{1 - u^2/c^2}}. \quad (4.19)$$

Time Dilation in Special Relativity

“Light Clock”

velocity = distance / time

speed of light = D / time



As seen by moving observer

As seen by stationary observer

Direction of motion

From the *Feynman Lectures*

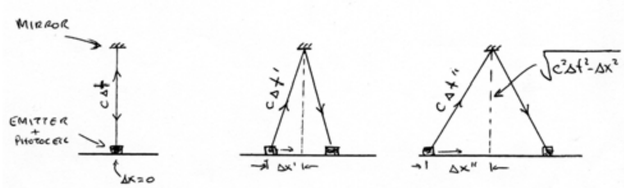
INTERVALS & METRICS

SPECIAL RELATIVITY

IF SPEED OF LIGHT SAME IN ALL REFERENCE FRAMES:
& LAWS OF PHYSICS SAME IN ALL INERTIAL FRAMES:

"CLOCK" THOUGHT-EXPERIMENT SHOWS:

1. TIME RUNS AT DIFFERENT SPEEDS AS SEEN FROM DIFFERENT REFERENCE FRAMES.
2. $c^2\Delta t^2 - \Delta x^2$ IS INVARIANT QUANTITY



CLOCK SEEN FROM ROCKET

SLOW-MOVING CLOCK SEEN FROM LAB

FAST-MOVING CLOCK SEEN FROM LAB

$$\text{INTERVAL}^2 = \Delta s^2 = c^2\Delta t^2 - \Delta x^2 = \text{INVARIANT.}$$

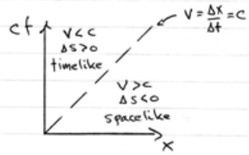
$\Updownarrow$

LORENTZ TRANSFORM

$$x' = \frac{x - ut}{\sqrt{1-u^2/c^2}}$$
$$y' = y$$
$$z' = z$$
$$t' = \frac{t - ux/c^2}{\sqrt{1-u^2/c^2}}.$$

SPACE-TIME (LORENTZ GEOMETRY)

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 = \text{INVARIANT}$$

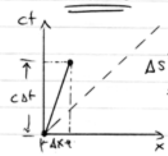


$\Delta s = 0$ for light

$0 = c^2\Delta t^2 - \Delta x^2$

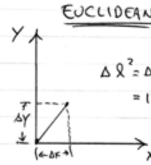
$v = \frac{\Delta x}{\Delta t} = c$

LORENTZ

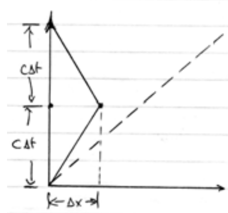


$\Delta s^2 = c^2\Delta t^2 - \Delta x^2$
= INVARIANT

EUCLIDEAN



$\Delta \ell^2 = \Delta x^2 + \Delta y^2$
= INVARIANT



CURVED PATHS ARE SHORTER INTERVALS IN LORENTZ GEOMETRY

STRAIGHT: $\Delta s = 2c\Delta t$

CURVED: $\Delta s = 2\sqrt{c^2\Delta t^2 - \Delta x^2}$

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2$$

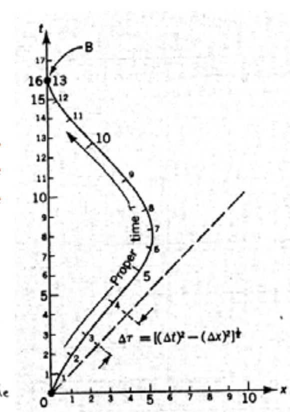
Generalize to: $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$

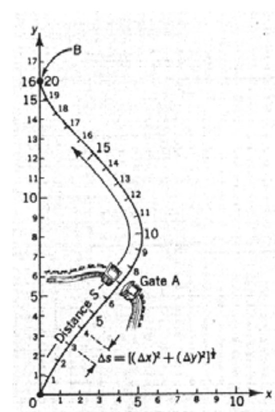
(Minkowski space)

World line: a particle's path through space-time

Proper time = $\int ds$ along time-like world-line

Proper time interval along world line is *smaller* than distance covered along time axis!





Euclidean geometry, for comparison.

Proper time = elapsed time on clock moving with particle

Proper distance = $\int\sqrt{-(ds^2)}$ along space-like world-line.

Figures from Taylor & Wheeler, "Spacetime Physics"

Metrics

From special relativity:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

For General Relativity:

$$ds^2 = \sum_{ij} g_{ij}\, dx^i dx^j \qquad g = \begin{bmatrix} c^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Field equation

$$G = -\frac{8\pi G}{c^4} T$$

Einstein Tensor: curvature of space. g_{ij} & derivatives.

mass + energy
Stress-energy tensor

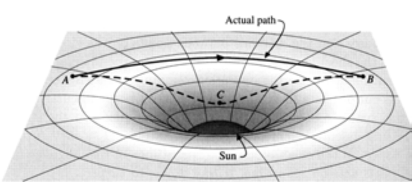
Schwarzschild metric (black holes, etc)

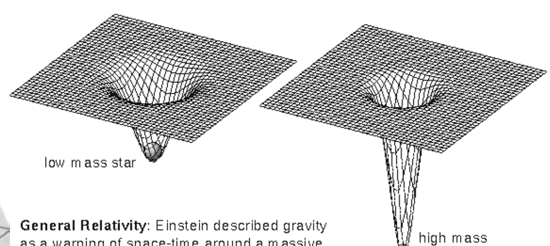
$$(ds)^2 = \left(c\,dt\sqrt{1-\frac{2GM}{rc^2}}\right)^2 - \left(\frac{dr}{\sqrt{1-\frac{2GM}{rc^2}}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\, d\phi)^2$$

Robertson-Walker metric (the Universe)

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\, d\theta)^2 + (\varpi \sin\theta\, d\phi)^2\right]$$

What curves into where?





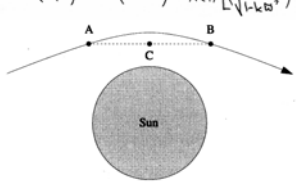
General Relativity: Einstein described gravity as a warping of space-time around a massive object. The stronger the gravity, the more space-time is warped.

Schwarzschild metric (black holes, etc)

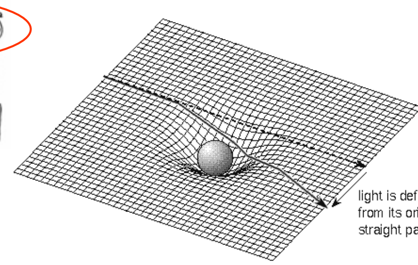
$$(ds)^2 = \left(c\,dt\sqrt{1-\frac{2GM}{rc^2}}\right)^2 - \left(\frac{dr}{\sqrt{1-\frac{2GM}{rc^2}}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\, d\phi)^2$$

Robertson-Walker metric (the Universe)

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\, d\theta)^2 + (\varpi \sin\theta\, d\phi)^2\right]$$



Objects follow shortest distance through curved space(-time).



General Relativity: Light travels along the curved space taking the shortest path between two points. Therefore, light is deflected toward a massive object! The stronger the local gravity is, the greater the light path is bent.

Paths of Objects through Curved Space-time

- *Geodesic* = straightest possible worldline
  - = maximum value of $\int ds$ in most examples.
  - But actually = extremum (max or min).
- Special Relativity:
  - Objects with no forces acting on them take straightest path through space–time.
  - Conservation of Energy-Momentum.
- General Relativity:
  - Objects follow Geodesics through curved space-time.
  - This follows directly from the equivalence principle.
- Light follows *null geodesic*
  - ds =0 at every point along path
  - ➔ $\int ds = 0$
  - ➔ d(length)/dt = c

Does the Schwarzschild metric actually tell us anything?

Now for some incredibly vigorous hand waving….

- The orbit of a satellite

Schwarzschild metric: $(ds)^2 = \left(c\,dt\sqrt{1-2GM/rc^2}\right)^2 - \left(\dfrac{dr}{\sqrt{1-2GM/rc^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$

Assume:
- Circular orbit ➔ $dr = 0$
- In plane where $d\phi = 0$
- Moving at constant angular velocity ω ➔ $d\theta = \omega\,dt$

$$(ds)^2 = \left[\left(c\sqrt{1-2GM/rc^2}\right)^2 - r^2\omega^2\right]dt^2 = \left(c^2 - \frac{2GM}{r} - r^2\omega^2\right)dt^2$$

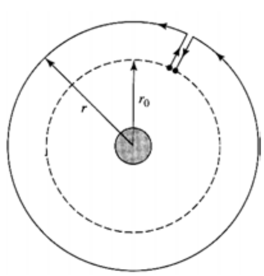
$$\Delta s = \int_0^{2\pi/\omega}\sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2}\,dt.$$

Find r that makes Δs be an extremum:

$$\frac{d}{dr}(\Delta s) = \frac{d}{dr}\left(\int_0^{2\pi/\omega}\sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2}\,dt\right) = 0$$

$$\frac{d}{dr}\sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2} = 0$$

$$\frac{2GM}{r^2} - 2r\omega^2 = 0$$



$$v = r\omega = \sqrt{\frac{GM}{r}}$$

A familiar Newtonian result!