The Schwarzschild radius

The Schwarzschild metric:

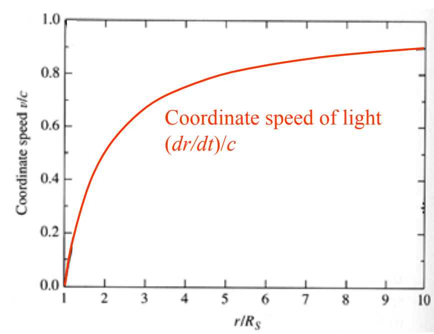
$$(ds)^2 = \left(c\,dt\sqrt{1-2GM/rc^2}\right)^2 - \left(\frac{dr}{\sqrt{1-2GM/rc^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$

- Things get funny at:

  $r = R_S = 2GM/c^2$

- To find the path of light: set $ds = 0$
- To make it easy, set $d\theta = d\phi = 0$
  ➔ light travelling radially

$$\frac{dr}{dt} = c\left(1 - \frac{2GM}{rc^2}\right) = c\left(1 - \frac{R_S}{r}\right)$$



Curved Spaces & the Robertson-Walker Metric

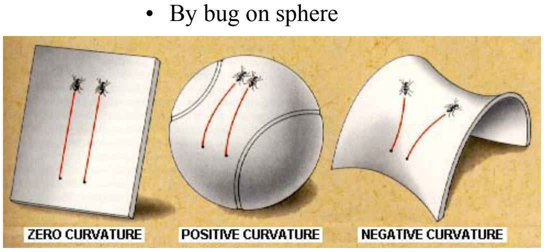
$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\,d\theta)^2 + (\varpi\sin\theta\,d\phi)^2\right]$$

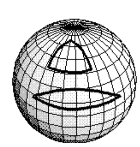
- R-W metric: most general solution for universe obeying Cosmological Principle.
  - Smooth distribution of matter.
  - Same everywhere.

    } ➔ Smooth curvature, same everywhere
  - Same everywhere at any given time.
- Curvature

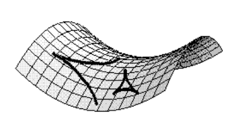
  $$K = \frac{1}{R^2}$$

  - Can be found from local measurements
    - By bug on sphere

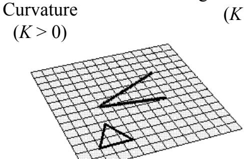




Positive Curvature ($K > 0$)



Negative Curvature ($K < 0$)



Flat ($K = 0$)

Geometry of a 2D Spherical Surface

$$r = R\sin\theta$$

$$dr = R\cos\theta\, d\theta$$

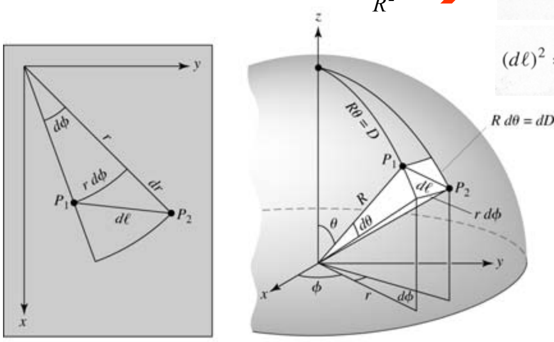
$$R\,d\theta = \frac{dr}{\cos\theta} = \frac{R\,dr}{\sqrt{R^2 - r^2}} = \frac{dr}{\sqrt{1 - r^2/R^2}} \rightarrow$$

$$(d\ell)^2 = (dD)^2 + (r\,d\phi)^2 = (R\,d\theta)^2 + (r\,d\phi)^2$$

$$(d\ell)^2 = \left(\frac{dr}{\sqrt{1 - r^2/R^2}}\right)^2 + (r\,d\phi)^2$$

$$K = \frac{1}{R^2} \rightarrow (d\ell)^2 = \left(\frac{dr}{\sqrt{1 - Kr^2}}\right)^2 + (r\,d\phi)^2$$

$$(d\ell)^2 = \left(\frac{dr}{\sqrt{1 - Kr^2}}\right)^2 + (r\,d\theta)^2 + (r\sin\theta\,d\phi)^2$$



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$$(d\ell)^2 = \left(\frac{dr}{\sqrt{1 - Kr^2}}\right)^2 + (r\,d\theta)^2 + (r\sin\theta\,d\phi)^2$$

To get R-W metric:

- Add time $$(ds)^2 = (c\,dt)^2 - \left(\frac{dr}{\sqrt{1 - Kr^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$
- Add another dimension $$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1 - k\varpi^2}}\right)^2 + (\varpi\,d\theta)^2 + (\varpi\sin\theta\,d\phi)^2\right]$$

where $r(t) = R(t)\varpi$ and $K(t) \equiv \frac{k}{R^2(t)}$

Dynamics of a 3D Surface (our Universe) in an Expanding 4D Space

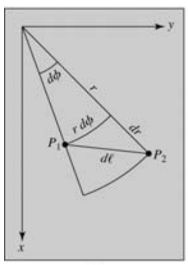
$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\,d\theta)^2 + (\varpi\sin\theta\,d\phi)^2\right]$$

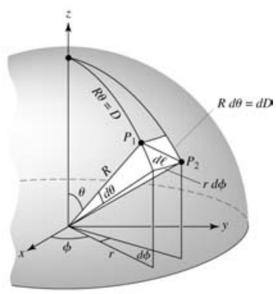
$$r(t) = R(t)\varpi$$

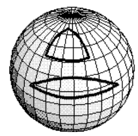
$$K(t) \equiv \frac{k}{R^2(t)}$$

$$\left(\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho\right)R^2 = -kc^2$$

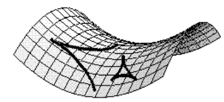
Friedmann Equation



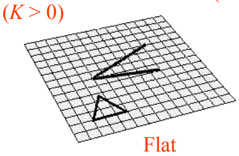




Positive Curvature ($K > 0$)



Negative Curvature ($K < 0$)



Flat ($K = 0$)

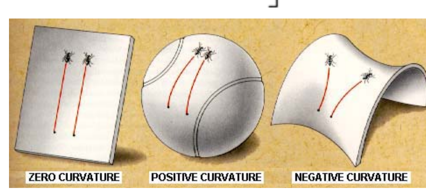
Dynamics of a 3D Surface (our Universe) in an Expanding 4D Space

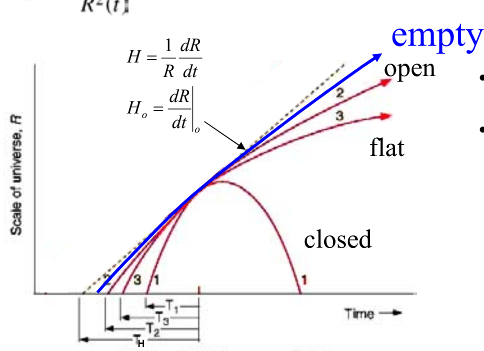
$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\,d\theta)^2 + (\varpi\sin\theta\,d\phi)^2\right]$$

$$r(t) = R(t)\varpi$$

$$K(t) \equiv \frac{k}{R^2(t)}$$

$$\left(\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho\right)R^2 = -kc^2$$





- Dynamics and curvature both due to mass-energy density.
- For Friedmann Eqn *without* Cosmological Constant:

Density	Curvature	Dynamics
$\rho_o < \rho_{c,o}$	Negative	Expands forever
$\rho_o = \rho_{c,o}$	Flat	Oozes to stop at $t = \infty$
$\rho_o > \rho_{c,o}$	Positive	Collapses back

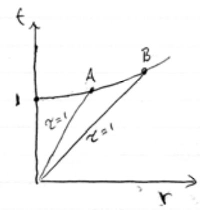
Why is the empty universe not flat??

Cosmological principle ➔ same age since Big Bang everywhere in the universe.

$$\tau^2 = t^2 - r^2 = (\text{Age of } U)^2 = 1$$

$$t = \sqrt{1 + r^2} \; ; \; t\,dt = r\,dr$$

If we are at $t = 1$, $r = 0$,
what is proper distance to A or B?



$$\int ds = \int (dr^2 - dt^2)^{1/2}$$

$$= \int \left(1 - \left(\frac{dt}{dr}\right)^2\right)^{1/2} dr$$

$$= \int \left(1 - \left(\frac{r}{t}\right)^2\right)^{1/2} dr$$

$$= \int \frac{\tau\, dr}{(\tau^2 + r^2)^{1/2}} = \int \frac{dr}{\sqrt{1 + \left(\frac{r}{\tau}\right)^2}}$$

For $r = R\tilde{\omega}$, $k = -\left(\frac{R}{\tau}\right)^2$

This is same as RW metric for open Universe $(k = -1)$

$$ds^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\tilde{\omega}}{\sqrt{1 - k\tilde{\omega}^2}}\right)^2 + \cdots\right]$$