Some expansions

http://mathworld.wolfram.com/SeriesExpansion.html

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots \text{ for } -1 < x < 1$$

$$\cos x = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \tfrac{1}{720}x^6 - \ldots \text{ for } -\infty < x < \infty$$

$$\cos^{-1} x = \tfrac{1}{2}\pi - x - \tfrac{1}{6}x^3 - \tfrac{3}{40}x^5 - \tfrac{5}{112}x^7 - \ldots \text{ for } -1 < x < 1$$

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \ldots \text{ for } -\infty < x < \infty$$

$$\sin x = x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 - \tfrac{1}{5040}x^7 + \ldots \text{ for } -\infty < x < \infty$$

$$\sin^{-1} x = x + \tfrac{1}{6}x^3 + \tfrac{3}{40}x^5 + \tfrac{5}{112}x^7 + \tfrac{35}{1152}x^9 + \ldots$$

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots \text{ for } -1 < x < 1$$

… and many, many more.

The Cosmological Constant

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\, d\theta)^2 + (\varpi \sin\theta\, d\phi)^2\right]$$

$$\left(\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho\right)R^2 = -kc^2$$

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2\right]R^2 = -kc^2$$

- Acts as outward force:

Newtonian version: $\frac{1}{2}mv^2 - G\frac{M_r m}{r} - \frac{1}{6}\Lambda m c^2 r^2 = -\frac{1}{2}mkc^2\varpi^2$

Define a potential:

$U_\Lambda \equiv -\frac{1}{6}\Lambda m c^2 r^2$ ➔ $\mathbf{F}_\Lambda = -\frac{\partial U_\Lambda}{\partial r}\hat{\mathbf{r}} = \frac{1}{3}\Lambda m c^2 r\,\hat{\mathbf{r}}$ [29.109]

$$\frac{d^2R}{dt^2} = \left\{-\frac{4}{3}\pi G\left[\rho_m + \rho_{\rm rel} + \frac{3(P_m + P_{\rm rel})}{c^2}\right] + \frac{1}{3}\Lambda c^2\right\}R$$ [29.112]

Psst… is it a constant?

Equation of state [CO pp. 1161-1162]

- Relation between P, R and ρ

Define: $\rho = \rho_o R^{-3(1+w)}$

Fluid eqn (29.50): $\frac{d(R^3\rho)}{dt} = -\frac{P}{c^2}\frac{d(R^3)}{dt}$ → Matter: $w = 0$ $\rho \propto R^{-3}$; Radiation: $w = 1/3$ $\rho \propto R^{-4}$; Cosm, Const: $w = -1$ $\rho \propto R^{0}$

[29.52] $P = wu = w\rho c^2$

Cosmological Constant as a "Negative Pressure"

Friedmann Eq. with Cosmological Constant: $\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2\right] R^2 = -kc^2$

$\rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G} = \text{constant} = \rho_{\Lambda,0}$ → $\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G(\rho_m + \rho_{rel} + \rho_\Lambda)\right] R^2 = -kc^2$ [29.114]

[29.111] $\frac{d(R^3\rho)}{dt} = -\frac{P}{c^2}\frac{d(R^3)}{dt}$ → $\frac{d^2R}{dt^2} = \left\{-\frac{4}{3}\pi G\left[\rho_m + \rho_{rel} + \frac{3(P_m + P_{rel})}{c^2}\right] + \frac{1}{3}\Lambda c^2\right\} R$

Negative pressure (29.115)

$P_\Lambda = -\rho_\Lambda c^2$ → $\frac{d^2R}{dt^2} = \left\{-\frac{4}{3}\pi G\left[\rho_m + \rho_{rel} + \rho_\Lambda + \frac{3(P_m + P_{rel} + P_\Lambda)}{c^2}\right]\right\} R$

[29.116]

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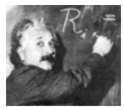
[29.116]

Psst… is it a constant?

$P = wu = w\rho c^2$ Is w really -1?

A slight renaming….

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2\right]R^2 = -kc^2$$



COSMOLOGICAL
CONSTANT

A slight renaming….

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2\right]R^2 = -kc^2 \qquad [29.108]$$

$$\rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G} = \text{constant} = \rho_{\Lambda,0} \qquad [29.113]$$

$$\left(\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G(\rho_m + \rho_{rel} + \rho_\Lambda)\right)R^2 = -kc^2$$

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda c^2}{3H^2}$$

[29.9]

$$\Omega \equiv \Omega_m + \Omega_{\text{rel}} + \Omega_\Lambda \qquad [29.119]$$

$$\left(H^2 - \frac{8}{3}\pi G\rho\right)R^2 = -kc^2 \rightarrow H^2(1-\Omega)R^2 = -kc^2$$

Friedmann Eqn

$$H_0^2(1-\Omega_0) = -kc^2 \qquad [29.121]$$

$$H_0^2(1-\Omega_0) = -kc^2$$ [29.121]

The basic WMAP result: $k = 0$

$$[\Omega_0]_{\text{WMAP}} = 1.02 \pm 0.02$$

$$\Omega_0 = \Omega_{m,0} + \Omega_{\text{rel},0} + \Omega_{\Lambda,0} = 1$$

$$[\Omega_{m,0}]_{\text{WMAP}} = 0.27 \pm 0.04$$

$$\Omega_{\text{rel},0} = 8.24 \times 10^{-5}$$

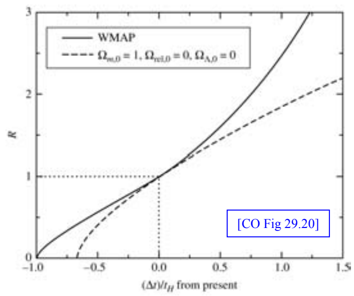
$$[\Omega_{\Lambda,0}]_{\text{WMAP}} = 0.73 \pm 0.04$$

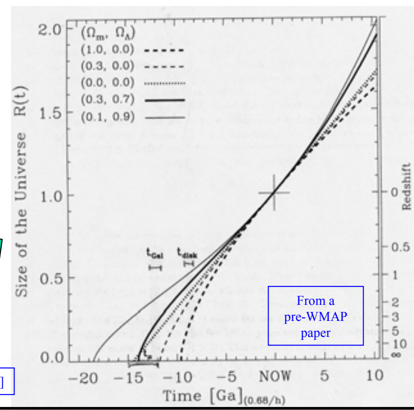
Still another form of Friedmann Eqn:

$$\left(\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\left(\rho_m + \rho_{rel} + \rho_\Lambda\right)\right)R^2 = -kc^2$$

Solution for $k = 0$:

$$t = \sqrt{\frac{3}{8\pi G}} \int_0^R \frac{R'\,dR'}{\sqrt{\rho_{m,0}R' + \rho_{\text{rel},0} + \rho_{\Lambda,0}R'^4}}.$$ [29.128]





Some Universes

Open *vs.* Closed:

$$k = 0 \;\Rightarrow\; \Omega_0 = \Omega_{m,0} + \cancel{\Omega_{\text{rel},0}}^{\,\sim 0} + \Omega_{\Lambda,0} = 1$$

Accelerating *vs.* Decelerating:

$$q(t) = -\frac{R(t)\,[d^2R(t)/dt^2]}{[dR(t)/dt]^2}$$

For $\Lambda = 0$: $\quad = \frac{1}{2}\Omega(t)$

$$q_0 = \frac{1}{2}\Omega_0$$

For $\Lambda = 0$:
$q_o = 0$ empty
< 0.5 open
$= 0.5$ flat
> 0.5 closed

For $\Lambda \neq 0$: $\quad q(t) = \frac{1}{2}\sum_i (1+3w_i)\Omega_i(t)$

$$q(t) = \frac{1}{2}\Omega_m(t) + \cancel{\Omega_{\text{rel}}}^{\,\sim 0}(t) - \Omega_\Lambda(t)$$

Expands Forever *vs.* Recollapses:

Does dR/dt ever = 0? See [29.135]