1. [5] Griffiths problem 4.19. We did much of this in class, but it’s important enough to warrant doing it again yourselves.

2. [15] In class, we solved the isotropic 2-dimensional harmonic oscillator problem by writing the Hamiltonian as a sum of x and y Hamiltonians:

\[
\hat{H} = \frac{\hat{P}_x^2 + \hat{P}_y^2}{2m} + \frac{1}{2} m \omega^2 \left( \hat{X}^2 + \hat{Y}^2 \right) = \hat{H}_x + \hat{H}_y
\]

with \( [\hat{H}_x, \hat{H}_y] = 0 \). We then found simultaneous eigenstates of \( \hat{H}_x \) and \( \hat{H}_y \), which obey

\[
\hat{H} |n_x, n_y\rangle = \left( n_x + n_y + 1 \right) \hbar \omega |n_x, n_y\rangle.
\]

In the position representation, these eigenstates are just products of x and y harmonic oscillator wavefunctions. Now I want you to find simultaneous eigenstates of \( \hat{H} \) and of the z-component of angular momentum: \( \hat{L}_z = \hat{X} \hat{P}_y - \hat{Y} \hat{P}_x \).

a) Show that \( [\hat{H}, \hat{L}_z] = 0 \), using Griffiths equation [4.10].

b) Define raising and lowering operators for the x and y the same way we did for the harmonic oscillator in one dimension:

\[
\hat{X} = \sqrt{\frac{\hbar}{2m \omega}} \left( a_x^+ + a_x \right) \quad \hat{P}_x = i \sqrt{\frac{\hbar m \omega}{2}} \left( a_x^+ - a_x \right)
\]

\[
\hat{Y} = \sqrt{\frac{\hbar}{2m \omega}} \left( a_y^+ + a_y \right) \quad \hat{P}_y = i \sqrt{\frac{\hbar m \omega}{2}} \left( a_y^+ - a_y \right)
\]

Express \( \hat{H} \) and \( \hat{L}_z \) in terms of these four new operators. (One answer is \( \hat{L}_z = \hbar (a_x a_y^+ - a_y a_x^+) \)).

c) Show that the ground state \( |0,0\rangle \) is an eigenstate of \( \hat{L}_z \). What is the eigenvalue?

d) Consider the 2-dimensional subspace of states with energy \( E = 2 \hbar \omega \), spanned by the old basis vectors \( |1,0\rangle \) and \( |0,1\rangle \). Find the eigenstates and eigenvalues of \( \hat{L}_z \) in this subspace. In other words, find two linear combinations of \( |1,0\rangle \) and \( |0,1\rangle \) that are eigenstates of \( \hat{L}_z \). Start by calculating \( \hat{L}_z |1,0\rangle \) and \( \hat{L}_z |0,1\rangle \), using Griffiths equation [2.66]. Now you may be able to write down the eigenstates of \( \hat{L}_z \) by inspection. If you can’t do that, then write the matrix form of \( \hat{L}_z \) in the old basis \( |1,0\rangle \) and \( |0,1\rangle \). Find the eigenvectors and eigenvalues of that matrix. Label your two new states \( |n = 1, m = \rangle \), where \( m \hbar \) is the eigenvalue of \( \hat{L}_z \). I suggest you keep the “\( n = \)” and “\( m = \)” inside the ket to distinguish these new states from the original basis states \( |n_x, n_y\rangle \).
e) Consider the 3-dimensional subspace of states with energy $E_2 = 3\hbar \omega$, spanned by the old basis vectors $|2,0\rangle$, $|1,1\rangle$, and $|0,2\rangle$. These states all have $n=2$, where $n=n_1 + n_2$. Do the same thing you did in part (d), but now you must write down the 3 x 3 matrix form of $\hat{L}_z$ in this subspace. If you have trouble with this part of the problem, skip it and go on to the next part.

f) What is the degeneracy of the $n^{th}$ harmonic oscillator energy level, with $E_n = (n + 1)\hbar \omega$? From your work on parts (d) and (e), guess which eigenvalues $m$ of $\hat{L}_z$ you would find in this subspace of states.

g) Write down the wavefunction for the state $|n = 1, m = 1\rangle = (|0,1\rangle - i|1,0\rangle)/\sqrt{2}$. In other words, write down $\psi(x, y) = \langle x, y|n = 1, m = 1\rangle$. (You don’t need to reinvent the wheel here; you are welcome to use Griffiths equations [2.59] and [2.62].) Now write down the same wavefunction in 2D polar coordinates, $\psi(r, \theta) = \langle r, \theta|n = 1, m = 1\rangle$. Use Euler’s formula to simplify your result. It should be obvious by inspection that $m=1$ for this state, based on the polar coordinate representation for the $z$-component of the angular momentum operator: $\hat{L}_z \rightarrow \frac{\hbar}{i} \frac{\partial}{\partial \theta}$. 