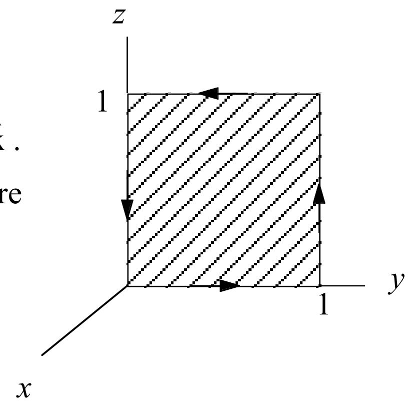
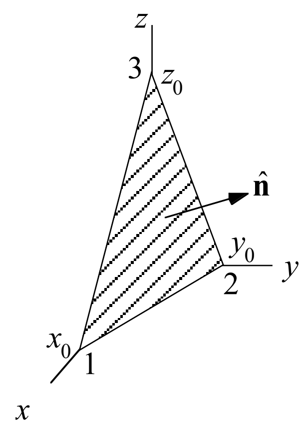
PHY481 Fall 2007 Practice Exam



1) [15 pts] Consider a vector function $\mathbf{F}(\mathbf{x}) = (2xy + 3y^2)\,\hat{\mathbf{j}} + (4yz^2)\,\hat{\mathbf{k}}$. Check Stokes theorem, $\oint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dA = \oint_C \mathbf{F} \cdot d\boldsymbol{\ell}$, over the square surface defined by the closed path shown in the figure.



2) [15 pts] To apply Gauss's law, $\oint_S \mathbf{E} \cdot d\mathbf{A} = q_{encl}/\varepsilon_0$, normal vectors at various points on a Gaussian surface are often needed.

a) [5 pts] The points $\mathbf{x}_1 = (x_0, 0, 0)$; $\mathbf{x}_2 = (0, y_0, 0)$; $\mathbf{x}_3 = (0, 0, z_0)$ define the planar surface as shown. Use a vector cross product to find the unit normal $\hat{\mathbf{n}}$ to the planar surface. Hint: the vectors must lie in the plane.

b) [5 pts] Determine the equation $z = f(x, y)$ satisfied by all points $\mathbf{x} = (x, y, z)$, that lie in the plane. Hint: given a point $\mathbf{x}'$ in the plane, the vectors $(\mathbf{x} - \mathbf{x}')$ lie in the plane, and $\hat{\mathbf{n}} \cdot (\mathbf{x} - \mathbf{x}') = 0$ for all $\mathbf{x}$.

c) [5 pts] A point charge q is located at the origin. Without an explicit calculation use Gauss's law to determine the electric flux through the surface shown.

3) [15 pts] Gauss's law states that integral of the electric flux through a closed surface is Q / ε_0, where Q is the enclosed charge.

a) [5 pts] For Gauss's law to hold, must the electric field in the neighborhood of the surface be determined only by the charge Q? Explain !

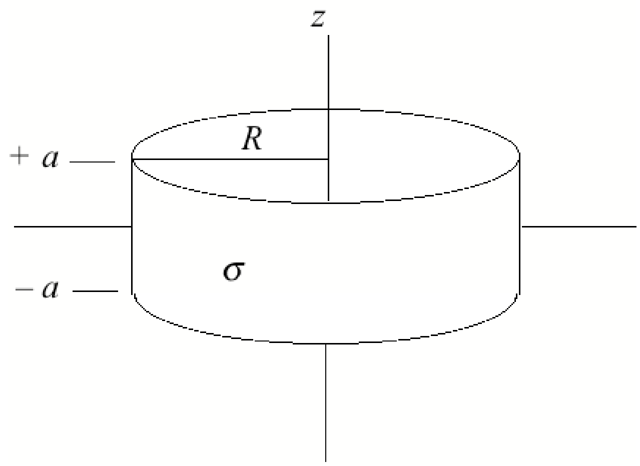
b) [10 pts] A region of space has a constant electric field $\mathbf{E} = E\hat{\mathbf{k}}$. An infinite sheet with surface charge density σ, is then *inserted* in the xy plane. What is the electric field vector above and below the sheet? Determine the total flux through a Gaussian surface cutting through the sheet (draw this situation). Compare this flux to the predictions of Gauss's law.

4) [15 pts] Consider a thin ring of charge with a surface charge density σ and height $2a$ centered on the xy plane.

a) [10 pts] Determine the electric field on the positive z axis.

b) [5 pts] For $z >> a, R$, show that the field approaches the field of a point charge with a value equal to the total charge on the ring. (Beware: this may be time consuming)

(Use approximations $(1+x)^n \approx 1+nx$ and $1/(1+x) \approx 1-x$, for $x << 1$.)



5) [15 pts] Use the Levi-Chivita tensor and the tensor product reduction to prove the identity $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$.
(If you have difficulty with this proof, instead prove $\mathbf{A} \times \mathbf{B} \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) + \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ to receive up to 8 pts.)

6) [15 pts] A point dipole $\mathbf{p} = p_0 \hat{\mathbf{k}}$ has the electric potential $V(r,\theta) = \dfrac{p_0 \cos\theta}{4\pi\varepsilon_0 r^2}$.

a) [10 pts] In spherical coordinates determine the electric field $\mathbf{E}(\mathbf{x})$.

b) [5 pts] Calculate the divergence of the field, $\nabla \cdot \mathbf{E}(\mathbf{x})$ (or $-\nabla^2 V$). Explain why it has the expected value.

7. [15 pts] Consider the following problems involving the Dirac delta function.

a) [10 pts] Evaluate the following integrals.

(i) $\int_2^6 dx(3x^2 - 2x - 1)\delta(x-3)$

(ii) $\int_{\text{all space}} d^3x \, (r^2 + \mathbf{r} \cdot \mathbf{r}_0 + r_0^2)\delta^3(\mathbf{r} - \mathbf{r}_0)$, where $\mathbf{r}_0$ is a constant vector

b) [5 pts] Use the Dirac delta function to define the charge density of an electric dipole, consisting of a point charge $-q$ at the origin and a point charge $+q$ at $\mathbf{x}_0$.