1) [15 pts] Consider a vector function \( \mathbf{F}(x) = (2xy + 3y^2) \hat{j} + (4yz^2) \hat{k} \).

Check Stokes theorem, \( \oint_{\partial S} \mathbf{F} \cdot d\mathbf{l} = \iint_{S} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dA \), over the square surface defined by the closed path shown in the figure.

2) [15 pts] To apply Gauss’s law, \( \oint_{S} \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\varepsilon_0} \), normal vectors at various points on a Gaussian surface are often needed.

a) [5 pts] The points \( \mathbf{x}_1 = (x_0, 0, 0); \mathbf{x}_2 = (0, y_0, 0); \mathbf{x}_3 = (0, 0, z_0) \) define the planar surface as shown. Use a vector cross product to find the unit normal \( \hat{n} \) to the planar surface. Hint: the vectors must lie in the plane.

b) [5 pts] Determine the equation \( z = f(x, y) \) satisfied by all points \( \mathbf{x} = (x, y, z) \), that lie in the plane. Hint: given a point \( \mathbf{x}' \) in the plane, the vectors \( \mathbf{x} - \mathbf{x}' \) lie in the plane, and \( \hat{n} \cdot (\mathbf{x} - \mathbf{x}') = 0 \) for all \( \mathbf{x} \).

c) [5 pts] A point charge \( q \) is located at the origin. Without an explicit calculation use Gauss’s law to determine the electric flux through the surface shown.

3) [15 pts] Gauss’s law states that integral of the electric flux through a closed surface is \( Q / \varepsilon_0 \), where \( Q \) is the enclosed charge.

a) [5 pts] For Gauss’s law to hold, must the electric field in the neighborhood of the surface be determined only by the charge \( Q \)? Explain!

b) [10 pts] A region of space has a constant electric field \( \mathbf{E} = E \hat{k} \). An infinite sheet with surface charge density \( \sigma \), is then inserted in the xy plane. What is the electric field vector above and below the sheet? Determine the total flux through a Gaussian surface cutting through the sheet (draw this situation). Compare this flux to the predictions of Gauss’s law.
4) [15 pts] Consider a thin ring of charge with a surface charge density $\sigma$ and height $2a$ centered on the xy plane.

a) [10 pts] Determine the electric field on the positive z axis.

b) [5 pts] For $z >> a, R$, show that the field approaches the field of a point charge with a value equal to the total charge on the ring. (Beware: this may be time consuming)

(Use approximations $(1 + x)^n \approx 1 + nx$ and $1/(1 + x) \approx 1 - x$, for $x \ll 1$.)

5) [15 pts] Use the Levi-Chivita tensor and the tensor product reduction to prove the identity \[ \nabla \times (A \times B) = A (\nabla \cdot B) - B (\nabla \cdot A) + (B \cdot \nabla) A - (A \cdot \nabla) B. \]

(If you have difficulty with this proof, instead prove \[ A \times B \times C = B (A \cdot C) + C (A \cdot B) \]
to receive up to 8 pts.)

6) [15 pts] A point dipole $\mathbf{p} = p_0 \hat{k}$ has the electric potential \[ V(r, \theta) = \frac{p_0 \cos \theta}{4 \pi \varepsilon_0 r^2}. \]

a) [10 pts] In spherical coordinates determine the electric field $\mathbf{E}(\mathbf{x})$.

b) [5 pts] Calculate the divergence of the field, $\nabla \cdot \mathbf{E}(\mathbf{x})$ (or $-\nabla^2 V$). Explain why it has the expected value.

7. [15 pts] Consider the following problems involving the Dirac delta function.

a) [10 pts] Evaluate the following integrals.

(i) $\int dx (3x^2 - 2x - 1) \delta(x - 3)$

(ii) $\int_{\text{all space}} d^3 x (r^2 + \mathbf{r} \cdot \mathbf{r}_0 + r_0^2) \delta^3 (\mathbf{r} - \mathbf{r}_0)$, where $\mathbf{r}_0$ is a constant vector

b) [5 pts] Use the Dirac delta function to define the charge density of an electric dipole, consisting of a point charge $-q$ at the origin and a point charge $+q$ at $\mathbf{x}_0$. 