PHY481: Electrostatics

Introductory E&M review (2)

Course web site: www.pa.msu.edu/courses/phy481

All homework is due AT 12:30 pm (no later)
HW1 due one week from today

Field of a charged spherical shell - Gauss's Law

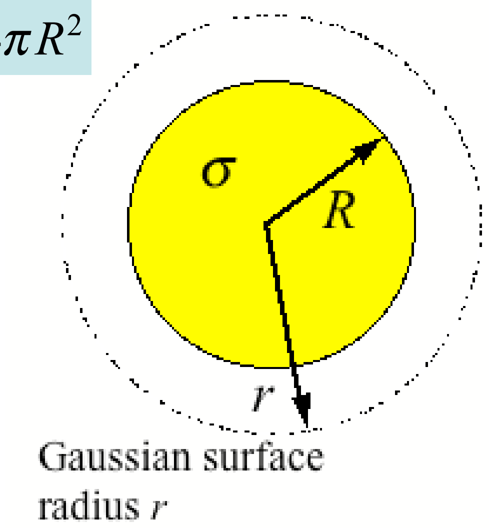
- Consider a radius R spherical shell with surface charge density σ.
  - A spherical Gaussian surface with radius $r < R$, is inside the charge surface, and encloses no charge -> $E_{inside} = 0$.
  - A spherical Gaussian surface with radius $r > R$, has the electric field normal to its surface.

$$\int_S \mathbf{E}\cdot d\mathbf{A} = E\int_0^{\pi} r^2 \sin\theta d\theta \int_0^{2\pi} d\phi$$

$$E4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$E = \frac{q}{4\pi\varepsilon_0 r^2} \;;\quad \mathbf{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$

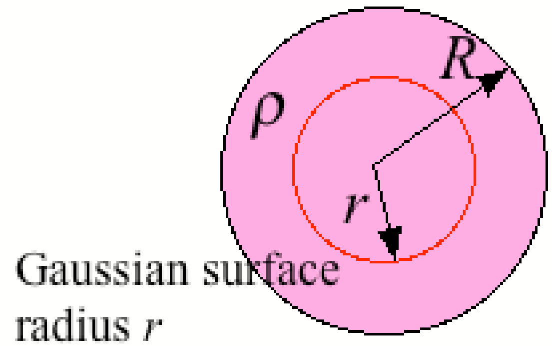
$$q = \sigma 4\pi R^2$$



Same electric field as charge q at the origin

Uniform charge density sphere - Gauss's Law

- Find the electric field inside of a sphere, radius *R*, with a uniform charge density ρ throughout the volume.
  - Pick a spherical Gaussian surface, radius r, inside the charged sphere



$$\int_S \mathbf{E}\cdot d\mathbf{A} = Er^2 \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$E 4\pi r^2 = \frac{q_{encl}}{\varepsilon_0} = \frac{Qr^3}{\varepsilon_0 R^3}$$

$$E = \frac{Qr}{4\pi\varepsilon_0 R^3} \; ; \quad \mathbf{E} = \frac{Qr}{4\pi\varepsilon_0 R^3}\hat{\mathbf{r}}$$

$$\rho = Q / \left(\tfrac{4}{3}\pi R^3\right)$$

$$q_{encl} = \rho\left(\tfrac{4}{3}\pi r^3\right) = Q\frac{r^3}{R^3}$$

  - Electric field outside of a sphere

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$

Same electric field as charge Q at the origin

Infinite sheet (again) - Gauss's Law

- Infinite sheet of charge with surface density σ.
  - Pick a cylindrical Gaussian surface, radius r, passing through the sheet.
  - The dot product **E**•d**A** is non zero only on TWO the ends.

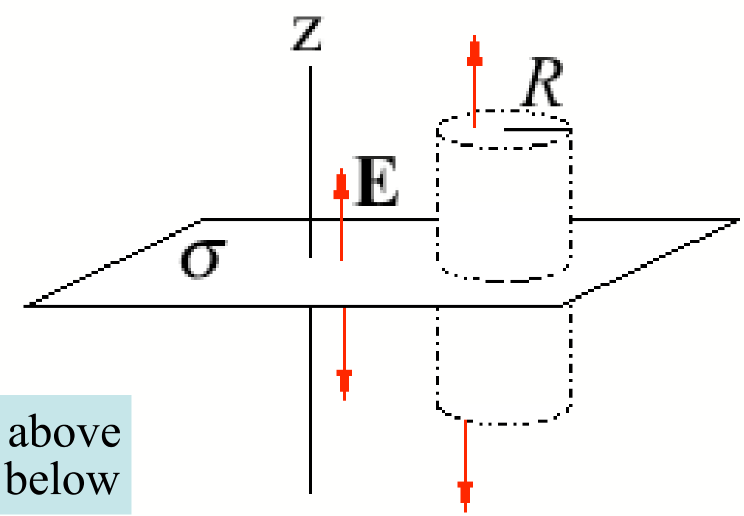
$$\int_S \mathbf{E}\cdot d\mathbf{A} = 2E\int_0^R r\,dr \int_0^{2\pi} d\phi$$

$$2(E\pi R^2) = \frac{q_{encl}}{\varepsilon_0} = \frac{\sigma(\pi R^2)}{\varepsilon_0}$$

$$E = \frac{\sigma}{2\varepsilon_0}; \quad \mathbf{E} = \pm\frac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}$$

+ above
– below

$$q_{encl} = \sigma(\pi R^2)$$



Electric Potential Energy $U(x)$

- Potential energy change ΔU of charge q' in known E-field.
  - Calculate the (negative) of the work done by the field along a path

$$\Delta U = -\int_{r_1}^{r_2} \mathbf{F}_{elec} \cdot d\mathbf{s} = -q' \int_{r_1}^{r_2} \mathbf{E} \cdot d\mathbf{s}$$

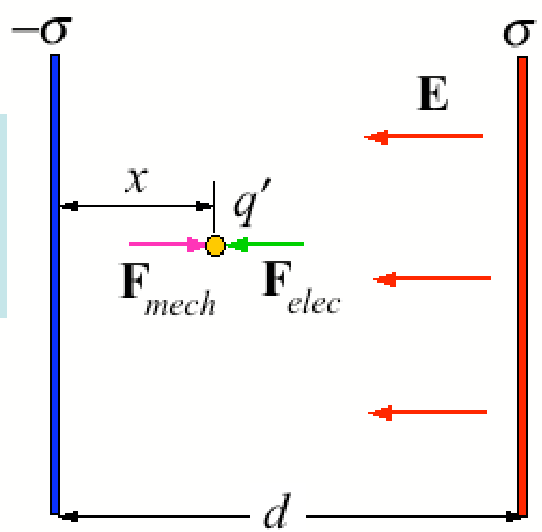
U is a scalar !

- Example: parallel plate capacitor

$$\Delta U = q'E\int_0^x dx' = q'Ex$$

$$\mathbf{E} = -E\,\hat{\mathbf{i}}$$

$$d\mathbf{s} = dx'\,\hat{\mathbf{i}}$$



- Move q' across entire capacitor

$$\Delta U = U(d) - U(0) = q'Ed$$

U = 0 at negative plate.

Electric potential V(x)

- Potential V(x) is potential energy/unit test charge *q′*
  - The potential *V* is defined without the test charge q′

U & V are scalars !

$$\Delta V = \frac{\Delta U}{q'} = -\int_{r_1}^{r_2} \mathbf{E} \cdot d\mathbf{s}$$

PE of test charge

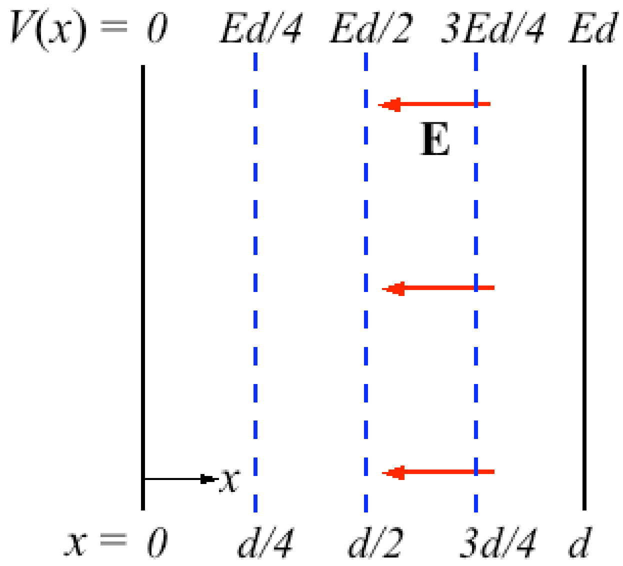
$$V(x) = \frac{U(x)}{q'}$$ + arbitary constant

- Example: parallel plate capacitor

Set V(0) = 0
constant = 0

$$V(x) = \frac{U(x)}{q'} = Ex$$

$$V(d) = Ed$$



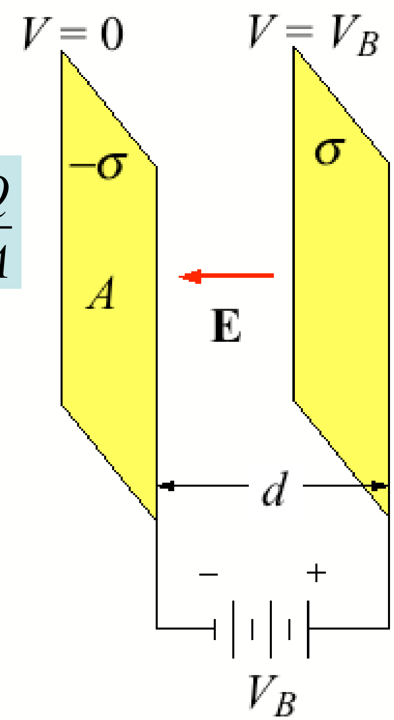
- Equipotential surfaces
  - Electric field lines always cross equipotential surfaces at 90°.
  - In parallel plate capacitors equipotenti surfaces are planes parallel to plates

Batteries, capacitors, and energy storage

- A battery moves charge Q between plates of area A
  - Battery moves electrons to create charge densities σ.
  - We have two expressions for electric field E !

$$E = \frac{V_B}{d} \qquad E = \frac{\sigma}{\varepsilon_0}$$

$$\sigma = \frac{Q}{A}$$



  - Find expression relating Q and V

$$Q = \frac{\varepsilon_0 A}{d} V_B = C V_B \qquad C = \frac{\varepsilon_0 A}{d}$$

  - Find energy stored while charging plates to Q

$$U = \int dU = \int_0^Q V dq = \int_0^Q \frac{q}{C} dq = \frac{Q^2}{2C}$$

$$U = \frac{1}{2} C V_B^2$$

Energy density

Show ! $u = \frac{1}{2}\varepsilon_0 E^2$

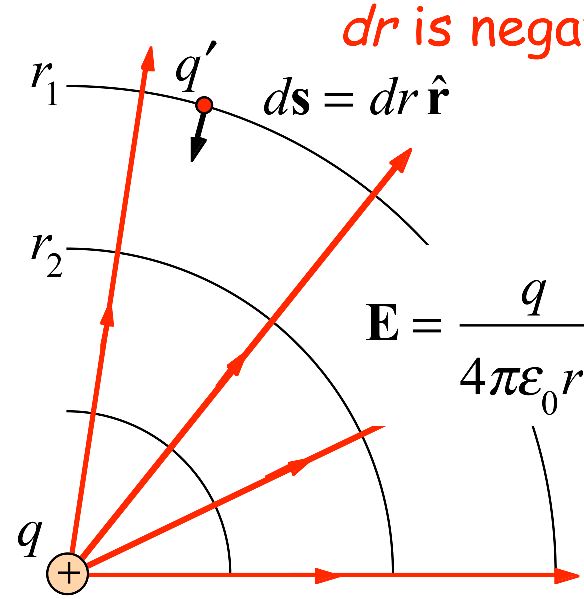
Potential of a point charge

- Move test charge q′ toward charge q.
  - ΔU is negative of work done by field in this motion

$$\Delta U = -\int_{r_1}^{r_2} \mathbf{F}_{elec} \cdot d\mathbf{s} = -\frac{qq'}{4\pi\varepsilon_0}\int_{r_1}^{r_2}\frac{dr}{r^2}$$

$$= \frac{qq'}{4\pi\varepsilon_0}\left[\frac{1}{r_2} - \frac{1}{r_1}\right] \quad (>0)$$

dr is negative



$$\Delta V = \frac{\Delta U}{q'} = V(r_2) - V(r_1)$$

- Potential of a point charge

V is a scalar !

$$V(r) = \frac{q}{4\pi\varepsilon_0 r}$$ $$V(\infty) = 0$$

$$\mathbf{E} = -\frac{dV}{dr}\hat{\mathbf{r}} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$

becomes -Grad (V)

Potential due to a charge distribution

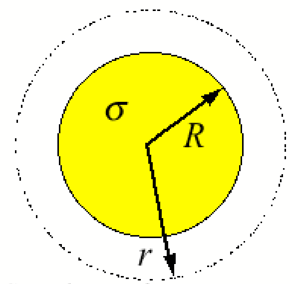
- Two ways to get potential of a charge distribution
  - Line integral of a known electric field
  - Integration of point charge potentials

$$\Delta V = -\int \mathbf{E}\cdot d\mathbf{s}$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r}$$

- Example: <u>spherical shell</u> with charge density σ
  - Electric field known from Gauss's Law
    (same as point charge r > R, and zero r < R)
  - Potential obtained from line integral along **E**

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$



$$\Delta V = -\frac{Q}{4\pi\varepsilon_0}\int_R^r \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r} - \frac{1}{R}\right]$$

outside point charge potential

$$V(r) = \frac{Q}{4\pi\varepsilon_0 r} \qquad r \geq R$$

$$V(r) = \frac{Q}{4\pi\varepsilon_0 R} \qquad r < R$$

inside potential is constant

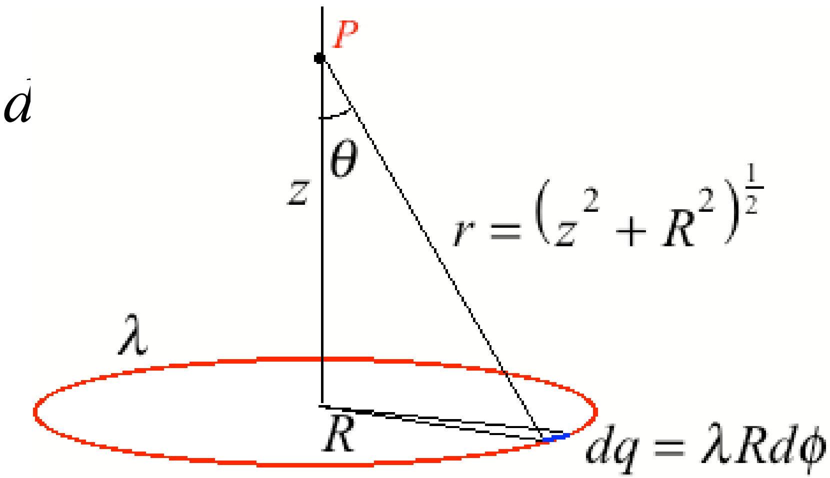
Potential by integration over point charges *dq*

- Potential on axis from ring with charge density λ

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r} = \frac{\lambda R}{4\pi\varepsilon_0 \left(z^2 + R^2\right)^{\frac{1}{2}}} \int_0^{2\pi} d$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(z^2 + R^2\right)^{-\frac{1}{2}}$$

$$\lambda = \frac{Q}{2\pi R}$$



- Obtain **E** on axis from (–)Gradient of *V*

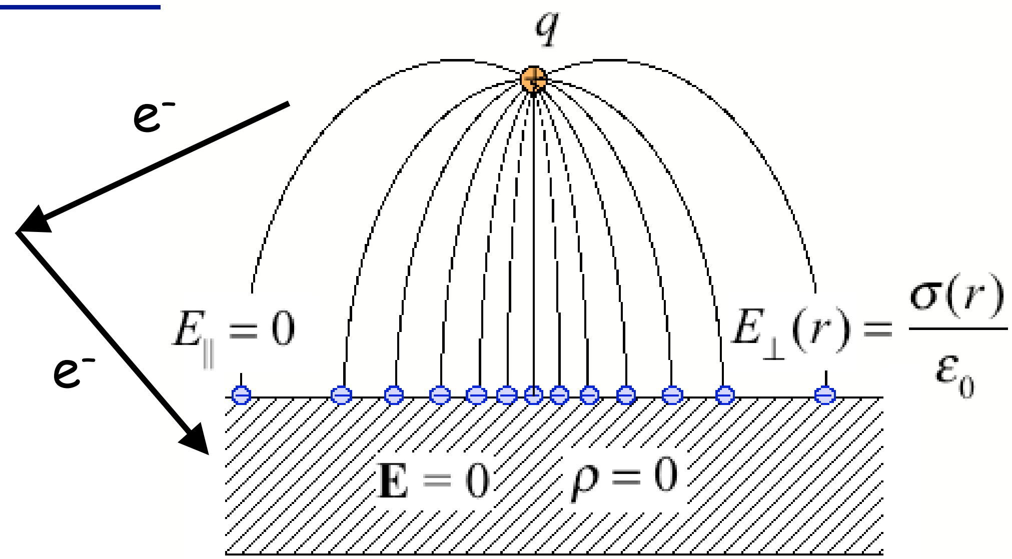
$$\mathbf{E} = -\frac{dV}{dz}\hat{\mathbf{k}} = \frac{Q}{4\pi\varepsilon_0} z\left(z^2 + R^2\right)^{-\frac{3}{2}}\hat{\mathbf{k}}$$

Note: $z\left(z^2 + R^2\right)^{-\frac{1}{2}} = \cos\theta$

- Why *V*? More tools available to determine *V* than **E**

Conductors and static electric fields

Move electrons from pebble to a metal surface. Pebble becomes charged +q.



- When charges stop moving, the electric field within the conductor is zero, charge is "pulled" to the surface. Also, Gauss's Law requires that the charge density within this conductor is zero.
- When charges stop moving, the components of the electric field parallel to the surface, $E_{||}$ = zero. Also, Gauss's Law requires that at the surface the electric field normal component, $E_{\perp} = \sigma / \varepsilon_0$.
- The electric potential is a constant throughout the conductor.
- Later we will learn "method of images" to determine fields & charges

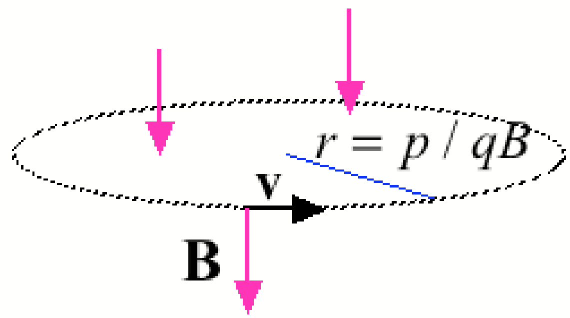
Magnetic fields

- A charge q moves at a velocity **v** in magnetic field **B**.
  - Force on the charge (use right hand rule for + charges)

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

  - Circular motion for **v** perpendicular to

$$F = mv^2 / r = qvB$$



- A current I flows in a thin wire
  - Force on small segment or on length

$$d\mathbf{F} = I\, d\boldsymbol{\ell} \times \mathbf{B} \qquad \mathbf{F} = I\,\mathbf{L} \times \mathbf{B}$$

  - Force between parallel straight wires

straight wire **B**

$$F = \frac{\mu_0 I_1 I_2 L}{4\pi d}; \quad \mu_0 = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/A} \longrightarrow \mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\phi}}$$

Attractive for same **I**'s

right hand rule

Magnetic fields from currents

- Ampere's Law
  - Closed path integral around current I

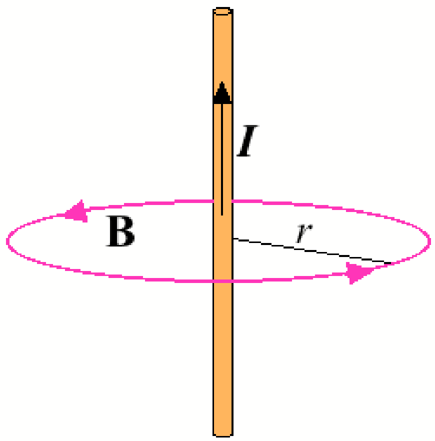
$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I$$

Choose a circular path!

  - Example: long straight wire carrying current I

$$\oint \mathbf{B} \cdot d\mathbf{s} = 2\pi r B = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r} \qquad \mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$



- Biot-Savart Law
  - Magnetic field from small current element d***I***

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\, d\ell \times \hat{\mathbf{r}}}{r^2}$$