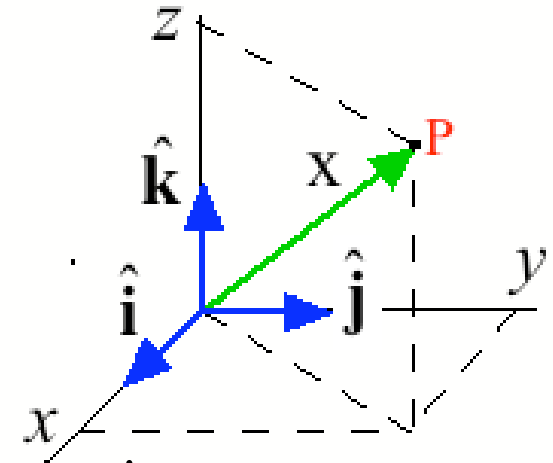

PHY481: Electromagnetism

Vector tools

Definitions

- Cartesian coordinates
 - Vector $\mathbf{x}$ is defined relative to the origin of primary coordinate system (x,y,z)
 - In Cartesian coordinates

$$\mathbf{x} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$



- Arbitrary rotation of the coordinate system
 - changes vector direction but not the magnitude

$$\mathbf{x}' = x'\hat{\mathbf{i}}' + y'\hat{\mathbf{j}}' + z'\hat{\mathbf{k}}'$$

$$|\mathbf{x}'| = |\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$$

- Example: rotation about the z-axis by angle ϕ

$$\hat{\mathbf{i}}' = \hat{\mathbf{i}} \cos \phi + \hat{\mathbf{j}} \sin \phi$$

$$\hat{\mathbf{j}}' = -\hat{\mathbf{i}} \sin \phi + \hat{\mathbf{j}} \cos \phi$$

$$\hat{\mathbf{k}}' = \hat{\mathbf{k}}$$

$$x' = \mathbf{x} \cdot \hat{\mathbf{i}}' = x \cos \phi + y \sin \phi$$

$$y' = \mathbf{x} \cdot \hat{\mathbf{j}}' = -x \sin \phi + y \cos \phi$$

$$z' = z$$

Rotation matrix

- A rotation can be expressed as a 3x3 **unitary** matrix R
 - Rotation of coordinates
 - Rotation of vector components

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix} = R \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

- Rotation Matrix is Unitary
 - Maintains vector magnitude
 - Transpose = Inverse

$$R_{ij}^T = R_{ji} = R_{ij}^{-1}$$

- Rotation matrix properties
 - $R = BCD$ (3 rotations)
 - Rotations do not commute

Euler angle rotations

$$B = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{rotation about } z$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix} \quad \text{rotation about } x'$$

$$D = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{rotation about } z'$$

Vector multiplication

- Vector addition and multiplication by a constant ✓
- Scalar (or dot) product

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z = \sum_{i=1}^3 A_i B_i \quad (1,2,3) = (x,y,z)$$

- Einstein summation convention (repeated indices)

$$\mathbf{A} \cdot \mathbf{B} = A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3 \quad \Sigma \text{ is superfluous}$$

- Cross product (new techniques)

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

$$C_i = A_j B_k - A_k B_j$$

i, j, k are cyclic permutations of 1,2,3

$$C_x = A_y B_z - A_z B_y$$

$$C_1 = A_2 B_3 - A_3 B_2$$

(1, 2, 3)

$$C_y = A_z B_x - A_x B_z$$

$$C_2 = A_3 B_1 - A_1 B_3$$

(1, 2, 3) → (2, 3, 1)

$$C_z = A_x B_y - A_y B_x$$

$$C_3 = A_1 B_2 - A_2 B_1$$

(2, 3, 1) → (3, 1, 2)

Economy of notation **we will NEED**

- Levi-Chivita anti-symmetric 3x3x3 tensor ε_{ijk}

Learn to love it ! $(\mathbf{A} \times \mathbf{B})_i = \varepsilon_{ijk} A_j B_k$ $i, j, k = (1, 2 \text{ or } 3)$

If any 2 indices are the same $\varepsilon_{ijk} = 0$ 21 of 27 elements are 0 !

Permutations of 123

6 non-zero elements

Cyclic = even # of pair-wise $\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = +1$

Non-cyclic = odd # of pair-wise $\varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1$

$$\begin{aligned} (\mathbf{A} \times \mathbf{B})_1 &= \varepsilon_{123} A_2 B_3 + \varepsilon_{132} A_3 B_2 \\ &= A_2 B_3 - A_3 B_2 \end{aligned}$$

Verify the remainder
for yourself

- Levi-Chivita tensor product

Kronecker delta

$$\varepsilon_{ijk} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

sum over k , $k = 1, 2, 3$

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Vector identities (Levi-Chivita tensor)

- Prove $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Cross product using Levi-Chivita tensor: $(\mathbf{A} \times \mathbf{B})_i = \varepsilon_{ijk} A_j B_k$

Levi-Chivita tensor product: $\varepsilon_{ijk} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$

Start proof here: $\mathbf{B} \times \mathbf{C} = \mathbf{D}$

i 'th component $[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})]_i = (\mathbf{A} \times \mathbf{D})_i = \varepsilon_{ijk} A_j D_k$ need indices l and m
 $D_k = (\mathbf{B} \times \mathbf{C})_k = \varepsilon_{klm} B_l C_m$

$$\begin{aligned} [\mathbf{A} \times (\mathbf{B} \times \mathbf{C})]_i &= \varepsilon_{ijk} \varepsilon_{klm} A_j B_l C_m \\ &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) A_j B_l C_m \\ &= A_j B_i C_j - A_j B_j C_i \quad A_j B_j = \mathbf{A} \cdot \mathbf{B} \end{aligned}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Vector operators - Gradient (Cartesian)

- Gradient operates on a scalar function V (potential)

Gives a vector in the direction of the maximum change in V

$$\begin{aligned}\nabla V(\mathbf{x}) &= \frac{\partial V}{\partial x_i} \hat{\mathbf{e}}_i \\ &= \frac{\partial V}{\partial x} \hat{\mathbf{i}} + \frac{\partial V}{\partial y} \hat{\mathbf{j}} + \frac{\partial V}{\partial z} \hat{\mathbf{k}}\end{aligned}$$

$$\nabla = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}$$

$$\begin{aligned}\hat{\mathbf{e}}_1 &= \hat{\mathbf{i}} \\ \hat{\mathbf{e}}_2 &= \hat{\mathbf{j}} \\ \hat{\mathbf{e}}_3 &= \hat{\mathbf{k}}\end{aligned}$$

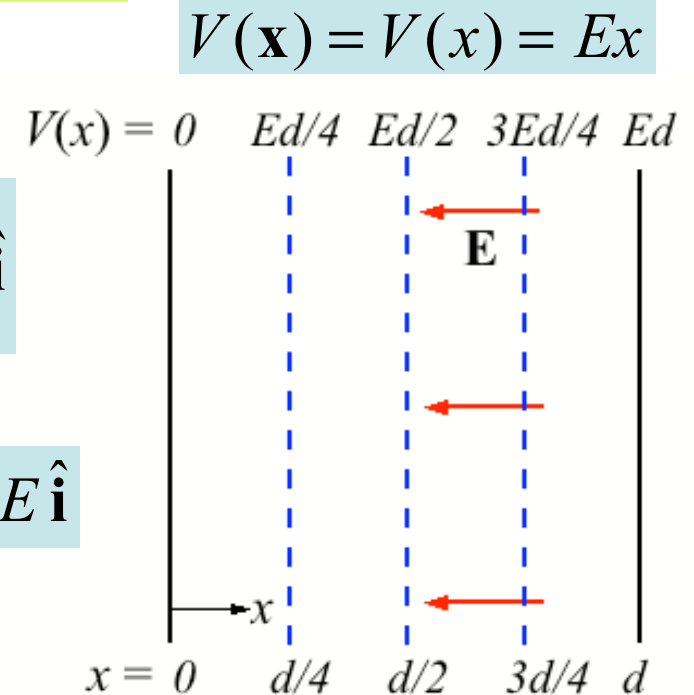
- Example: parallel plate capacitor

Maximum change in V is in the **+x direction**

$$\nabla V(\mathbf{x}) = \frac{\partial V}{\partial x} \hat{\mathbf{i}} = E \hat{\mathbf{i}}$$

Electric field **E** points in the direction opposite to maximum change in V , -x direction

$$\mathbf{E}(\mathbf{x}) = -\nabla V(\mathbf{x}) = -E \hat{\mathbf{i}}$$



Vector operators - Divergence (Cartesian)

- Divergence of a vector function is a scalar

Del operator

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = \frac{\partial E_i}{\partial x_i} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

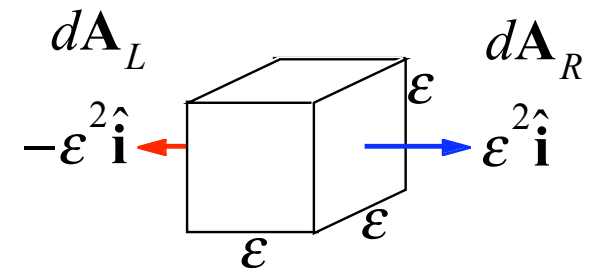
Spreading of $\mathbf{E}$ at $\mathbf{x}$.
 $\neq 0$ if charge is at $\mathbf{x}$

$$\nabla = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}$$

- Coordinate independent definition

tiny box centered on $\mathbf{x}$

$$\begin{aligned} \oint_S \mathbf{E}(\mathbf{x}) \cdot d\mathbf{A} &= \sum_{i=1}^3 \left[E_i(\mathbf{x} + \frac{\epsilon}{2} \hat{\mathbf{e}}_i) \epsilon^2 - E_i(\mathbf{x} - \frac{\epsilon}{2} \hat{\mathbf{e}}_i) \epsilon^2 \right] \\ &= \sum_{i=1}^3 \left[\frac{E_i(\mathbf{x} + \frac{\epsilon}{2} \hat{\mathbf{e}}_i) - E_i(\mathbf{x} - \frac{\epsilon}{2} \hat{\mathbf{e}}_i)}{\epsilon} \right] \epsilon^3 \\ &= \sum_{i=1}^3 \left[\frac{\partial E_i}{\partial x_i} \right] \epsilon^3 = [\nabla \cdot \mathbf{E}(\mathbf{x})] \epsilon^3 \end{aligned}$$



Divergence definition

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = \lim_{V \rightarrow 0} \frac{1}{V} \oint_S \mathbf{E}(\mathbf{x}) \cdot d\mathbf{A}$$

Differential form of Gauss's Law

- Gauss's Law in terms of divergence of $\mathbf{E}$

Gauss's Law

$$\oint_S \mathbf{E}(\mathbf{x}) \cdot d\mathbf{A} = \frac{q_{encl}}{\epsilon_0}$$

$$\frac{1}{V} \oint_S \mathbf{E}(\mathbf{x}) \cdot d\mathbf{A} = \frac{1}{V} \frac{q_{encl}}{\epsilon_0}$$

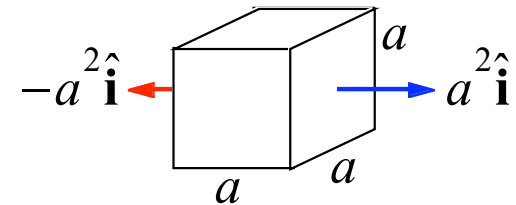
$$\lim_{V \rightarrow 0} \frac{1}{V} \oint_S \mathbf{E}(\mathbf{x}) \cdot d\mathbf{A} = \lim_{V \rightarrow 0} \frac{q_{encl}}{V} \frac{1}{\epsilon_0}$$

$$\lim_{V \rightarrow 0} \frac{1}{V} \oint_S \mathbf{E}(\mathbf{x}) \cdot d\mathbf{A} = \nabla \cdot \mathbf{E}(\mathbf{x})$$

Gauss's Law
equivalent

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\epsilon_0}$$

Box centered on $\mathbf{x}$



$$\rho = \frac{q_{encl}}{V}; \quad V = a^3$$

see previous slide

Vector operators - Curl (Cartesian)

- Curl of a vector function is another vector

$$[\nabla \times \mathbf{B}(\mathbf{x})]_i = \varepsilon_{ijk} \frac{\partial B_k}{\partial x_j}$$

"Circulation" of $\mathbf{B}$ around a loop

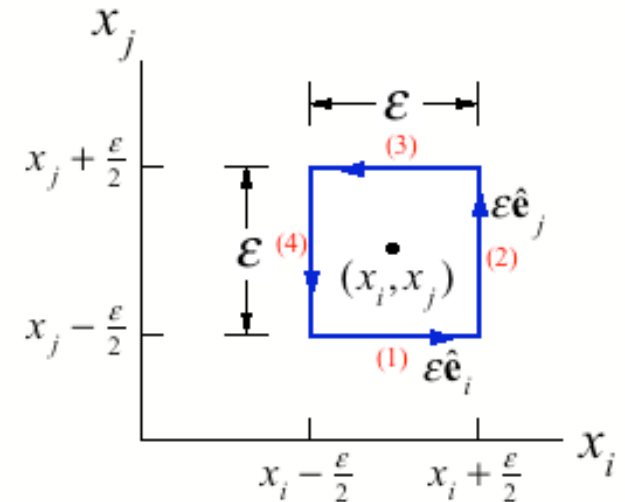
$$(\nabla \times \mathbf{B}(\mathbf{x}))_1 = \frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3}$$

Verify the remainder

- Coordinate independent definition

$$\begin{aligned} \oint_{P_k} \mathbf{B}(\mathbf{x}) \cdot d\ell &= \left[B_i(\mathbf{x} - \frac{\varepsilon}{2} \hat{\mathbf{e}}_j) \varepsilon - B_i(\mathbf{x} + \frac{\varepsilon}{2} \hat{\mathbf{e}}_j) \varepsilon \right] \\ &\quad + \left[B_j(\mathbf{x} + \frac{\varepsilon}{2} \hat{\mathbf{e}}_i) \varepsilon - B_j(\mathbf{x} - \frac{\varepsilon}{2} \hat{\mathbf{e}}_i) \varepsilon \right] \\ &= \left[\frac{\partial B_j}{\partial x_i} - \frac{\partial B_i}{\partial x_j} \right] \varepsilon^2 = [\nabla \times \mathbf{B}(\mathbf{x})]_k \varepsilon^2 \end{aligned}$$

Curl de



$$\hat{\mathbf{n}} \cdot [\nabla \times \mathbf{B}(\mathbf{x})] = \lim_{A \rightarrow 0} \frac{1}{A} \oint_C \mathbf{B}(\mathbf{x}) \cdot d\ell$$