PHY481: Electromagnetism

Del identities & integral theorems

Vector operators - Curl (Cartesian)

- Curl of a vector function is another vector

$$[\nabla \times \mathbf{B}(\mathbf{x})]_i = \varepsilon_{ijk} \frac{\partial B_k}{\partial x_j}$$

"Circulation" of **B** around a loop

$$(\nabla \times \mathbf{B}(\mathbf{x}))_1 = \frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3}$$

Verify the remainder

Closed path

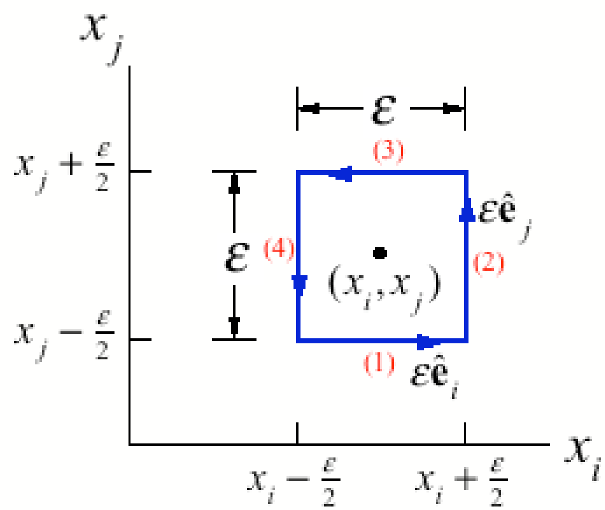
- Coordinate independent definition

$$\oint_{C_k} \mathbf{B}(\mathbf{x}) \cdot d\ell = \left[B_i(\mathbf{x} \overset{(1)}{-} \tfrac{\varepsilon}{2}\hat{\mathbf{e}}_j)\varepsilon - B_i(\mathbf{x} \overset{(3)}{+} \tfrac{\varepsilon}{2}\hat{\mathbf{e}}_j)\varepsilon \right]$$

$$+ \left[B_j(\mathbf{x} \overset{(2)}{+} \tfrac{\varepsilon}{2}\hat{\mathbf{e}}_i)\varepsilon - B_j(\mathbf{x} \overset{(4)}{-} \tfrac{\varepsilon}{2}\hat{\mathbf{e}}_i)\varepsilon \right]$$

$$= \left[\underset{(2\&4)}{\frac{\partial B_j}{\partial x_i}} - \underset{(1\&3)}{\frac{\partial B_i}{\partial x_j}} \right] \varepsilon^2 = [\nabla \times \mathbf{B}(\mathbf{x})]_k \, \varepsilon^2$$

C_k : closed path in plane $\perp$ to $\hat{\mathbf{e}}_k$



Curl de

$$\hat{\mathbf{n}} \cdot [\nabla \times \mathbf{B}(\mathbf{x})] = \lim_{A \to 0} \frac{1}{A} \oint_C \mathbf{B}(\mathbf{x}) \cdot d\ell$$

Cartesian vector operators (Einstein notation)

Dot Product

$$\mathbf{A}\cdot\mathbf{B} = A_i B_i$$

Levi-Chivita Tensor

$$\varepsilon_{ijk}$$

Cross Product

$$(\mathbf{A}\times\mathbf{B})_i = \varepsilon_{ijk} A_j B_k$$

Permutations of 123

6 non-zero elements

Cyclic = even # of pair-wise $\quad \varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = +1$

Non-cyclic = odd # of pair-wise $\quad \varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1$

Tensor product

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Kronecker delta

$$\delta_{ij} = \begin{cases} 1\,,\ i = j \\ 0\,,\ i \neq j \end{cases}$$

Del Operator

$$\nabla = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}$$

$$\hat{\mathbf{e}}_1 = \hat{\mathbf{i}},\ \hat{\mathbf{e}}_2 = \hat{\mathbf{j}},\ \hat{\mathbf{e}}_3 = \hat{\mathbf{k}}$$

Gradient

$$\nabla V(\mathbf{x}) = \frac{\partial V}{\partial x_i}\hat{\mathbf{e}}_i$$

Divergence

$$\nabla\cdot\mathbf{E}(\mathbf{x}) = \frac{\partial E_i}{\partial x_i}$$

Curl

$$[\nabla\times\mathbf{B}(\mathbf{x})]_i = \varepsilon_{ijk}\frac{\partial B_k}{\partial x_j}$$

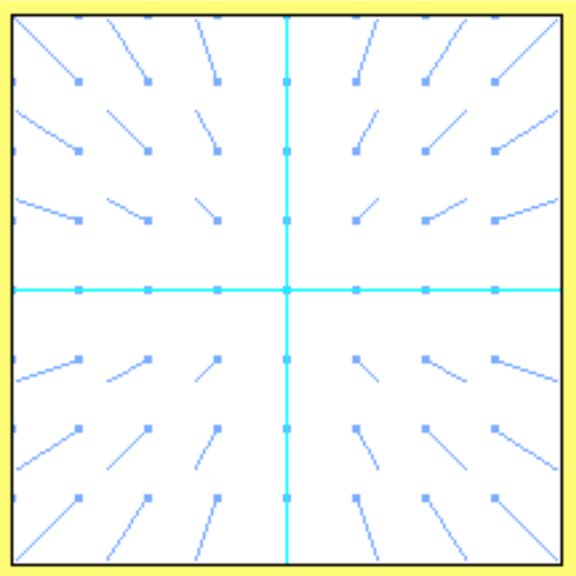
$$\nabla\cdot\nabla V = \nabla^2 V$$

Laplacian of scalar function *V*

Physical interpretation of vector operators

- Characterize "flow" of field [area displayed (±3m, ±3m)]

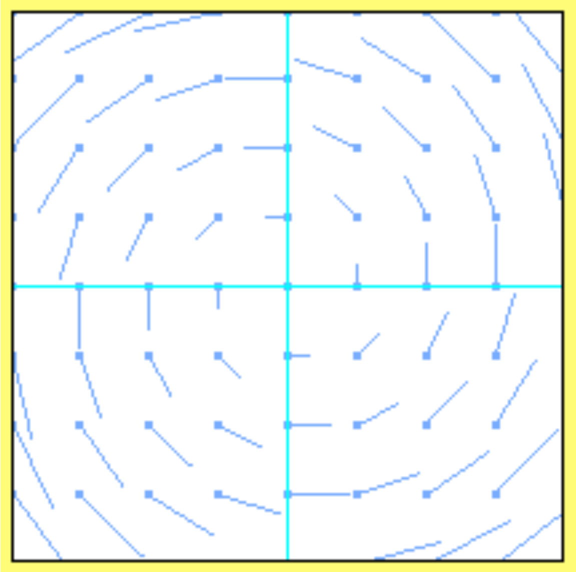
$\mathbf{E}(\mathbf{x}) = \left(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}\right)\ \mathrm{V/m^2}$



$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{x})}{\varepsilon_0} = 2\ \mathrm{V/m^2}$$

$\nabla \times \mathbf{E} = 0$ E is "irrotational"

$\mathbf{B}(\mathbf{x}) = \left(-y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}\right)\ \mathrm{T/m^2}$

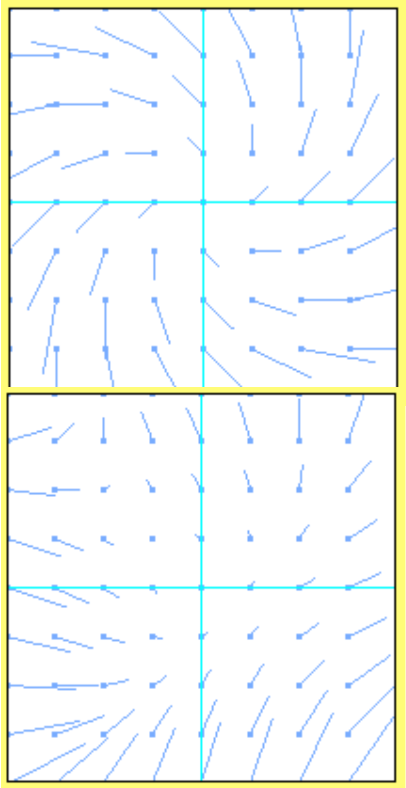


$\nabla \cdot \mathbf{B} = 0$ B is "solenoidal"

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}(\mathbf{x}) = 2\hat{\mathbf{k}}\ \mathrm{T/m^2}$$

$\nabla \times \mathbf{C} \neq 0$

$\nabla \cdot \mathbf{C} \neq 0$



(see http://www.math.gatech.edu/~carlen/2507/notes/vectorCalc/dcvisualize.html)

Del identity

- Prove $\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}$

Let $\mathbf{G}(\mathbf{x}) = \nabla\times\mathbf{F}(\mathbf{x})$

Similar to **Ax(BxC)** Identity

$$[\nabla\times(\nabla\times\mathbf{F})]_i = [\nabla\times\mathbf{G}(\mathbf{x})]_i = \varepsilon_{ijk}\frac{\partial G_k}{\partial x_j}$$

$$= \varepsilon_{ijk}\varepsilon_{klm}\frac{\partial^2 F_m}{\partial x_j \partial x_l}$$

$$= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\frac{\partial^2 F_m}{\partial x_j \partial x_l}$$

$$= \frac{\partial}{\partial x_i}\left(\frac{\partial F_j}{\partial x_j}\right) - \frac{\partial^2 F_i}{\partial x_j \partial x_j}$$

Replace

$$G_k = \varepsilon_{klm}\frac{\partial F_m}{\partial x_l}$$

$$\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}$$

A tougher Del identity

- Prove $$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

Don't start with left hand side ! Expansion will be too hard.

$$[\mathbf{A}\times(\nabla\times\mathbf{B})]_i = \varepsilon_{ijk} A_j \left(\varepsilon_{klm} \frac{\partial B_m}{\partial x_l}\right)$$

$$= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) A_j \frac{\partial B_m}{\partial x_l}$$

$$= A_j \frac{\partial B_j}{\partial x_i} - A_j \frac{\partial B_i}{\partial x_j}$$

$$[\mathbf{B}\times(\nabla\times\mathbf{A})]_i = B_j \frac{\partial A_j}{\partial x_i} - B_j \frac{\partial A_i}{\partial x_j}$$

$$[(\mathbf{A}\cdot\nabla)\mathbf{B}]_i = A_j \frac{\partial B_i}{x_j}$$

$$[(\mathbf{B}\cdot\nabla)\mathbf{A}]_i = B_j \frac{\partial A_i}{x_j}$$

$$[\nabla(\mathbf{A}\cdot\mathbf{B})]_i = \frac{\partial A_j}{\partial x_i} B_j + A_j \frac{\partial B_j}{\partial x_i}$$

Not so tough (Problem 2.22)

- Given $h(\mathbf{x}) = (\mathbf{x}\times\mathbf{A})\cdot(\mathbf{x}\times\mathbf{B})$ — $\mathbf{A}\ \&\ \mathbf{B}$ are constants, $\mathbf{x} = x_i\hat{\mathbf{e}}_i$
- Prove $\nabla h(\mathbf{x}) = \mathbf{A}\times(\mathbf{x}\times\mathbf{B}) + \mathbf{B}\times(\mathbf{x}\times\mathbf{A})$

$$h(\mathbf{x}) = (\mathbf{x}\times\mathbf{A})\cdot(\mathbf{x}\times\mathbf{B}) = \left(\varepsilon_{ijk}x_jA_k\right)\left(\varepsilon_{ilm}x_lB_m\right)$$

$$= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})x_jx_lA_kB_m = x_j^2A_kB_k - x_jB_jx_kA_k$$

$$[\nabla h(\mathbf{x})]_i = \frac{\partial}{\partial x_i}\left(x_j^2A_kB_k - x_jB_jx_kA_k\right)$$

$$= 2x_iA_kB_k - B_ix_kA_k - x_jB_jA_i$$

$$\nabla h(\mathbf{x}) = 2\mathbf{x}(\mathbf{A}\cdot\mathbf{B}) - \mathbf{B}(\mathbf{x}\cdot\mathbf{A}) - \mathbf{A}(\mathbf{x}\cdot\mathbf{B})$$

$$= \left[\mathbf{x}(\mathbf{A}\cdot\mathbf{B}) - \mathbf{B}(\mathbf{A}\cdot\mathbf{x})\right] + \left[\mathbf{x}(\mathbf{A}\cdot\mathbf{B}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{x})\right]$$

(lecture 3)

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$$

$$\nabla h(\mathbf{x}) = \mathbf{A}\times(\mathbf{x}\times\mathbf{B}) + \mathbf{B}\times(\mathbf{x}\times\mathbf{A})$$

Potential enabling identities

- Electric scalar potential $f = V$ follows from $\nabla \times (\nabla f) = 0$

$$[\nabla \times (\nabla f)]_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_k}\right) = 0$$

$$\varepsilon_{ijk} \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_k} = -\varepsilon_{ikj} \frac{\partial}{\partial x_k}\frac{\partial f}{\partial x_j} = -\varepsilon_{ijk} \frac{\partial^2 f}{\partial x_j \partial x_k} = 0$$

Partials can be taken in any order

- Magnetic vector potential **F** = **A** follows from $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial}{\partial x_i}\left(\varepsilon_{ijk} \frac{\partial F_k}{\partial x_j}\right) = \varepsilon_{ijk} \frac{\partial^2 F_k}{\partial x_i \partial x_j} = 0$$

ε_{ijk} is antisymmetric, and $\dfrac{\partial^2 F_k}{\partial x_i \partial x_j}$ symmetric

Some double Dels ≠ 0

$$\nabla \times (\nabla \times \mathbf{F}) \neq 0$$

$$\nabla (\nabla \cdot \mathbf{F}) \neq 0$$

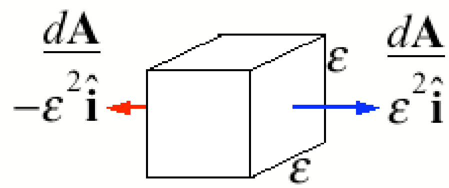
Gauss's Theorem

- Divergence definition

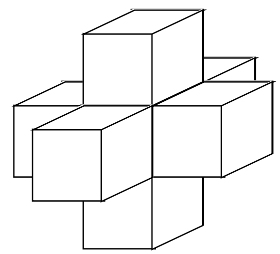
$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^3} \oint_S \mathbf{E} \cdot d\mathbf{A} = \nabla \cdot \mathbf{E}$$

$$\mathbf{E} \cdot d\mathbf{A}$$

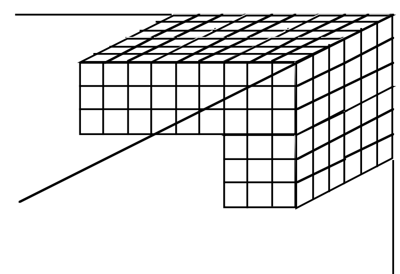
"flux" of **E** through d**A**



- "Proof" of Gauss's Theorem
  - Place 6 tiny cubes against the original cube.
  - On adjacent faces there is a common **E**, but an opposite sign for each d**A**, thus **E**•d**A** cancels.
  - Summing over a block broken into tiny cubes, the surface fluxes cancel except on the block's surface.



$$\sum_{i=1}^{N} \left[\oint_{S_i} \mathbf{E} \cdot d\mathbf{A} \right] = \sum_{i=1}^{N} \left[\nabla \cdot \mathbf{E}(\mathbf{x}_i) \right] \varepsilon^3$$



- Gauss's Theorem for a closed surface S

Only the external surface flux left

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \oint_V \nabla \cdot \mathbf{E}(\mathbf{x})\, d^3x$$

Divergence integrated over the volume inside S

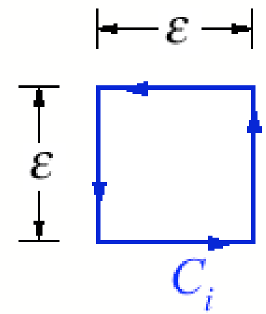
Stokes's theorem

- Curl definition

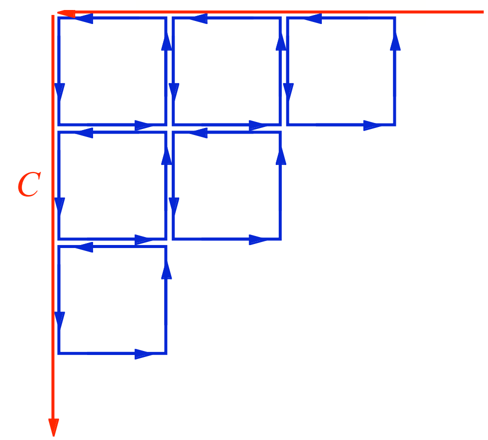
$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \oint_C \mathbf{B} \cdot d\ell = [\nabla \times \mathbf{B}] \cdot \hat{\mathbf{n}}$$

$\mathbf{B} \cdot d\ell$ "line element of **B** along dl"

$\hat{\mathbf{n}}$ "normal to surface bounded by curve"



- "Proof" of Stokes's Theorem
  - Break large loop C into many tiny loops.
  - On adjacent sides there is a common **B** but an opposite sign for each dl thus line elements cancel.
  - The tiny loop line elements cancel except those on the large loop.



$$\sum_{i=1}^{N} \left[\oint_{C_i} \mathbf{B} \cdot d\ell \right] = \sum_{i=1}^{N} \left[\nabla \times \mathbf{B}(\mathbf{x}_i) \right] \cdot \hat{\mathbf{n}} \varepsilon^2$$

- Stokes's Theorem for a closed curve *C*

Only line elements on *C* remain

$$\oint_C \mathbf{B} \cdot d\ell = \oint_S \left[\nabla \times \mathbf{B}(\mathbf{x}) \right] \cdot d\mathbf{A}$$

Curl integrated over <u>any</u> surface S bounded by C

Summary of integral theorems

- Gauss's Theorem

Flux integrated over closed surface S = Divergence integrated over the volume inside S

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \oint_V \nabla \cdot \mathbf{E}(\mathbf{x})\, d^3x$$

Neither side looks particularly simple, however, for static electric fields the right hand side will be shown to be equal to the enclosed charge/ε_0

- Stokes's Theorem

Line elements integrated over closed curve = Curl integrated over any surface S bounded by C

$$\oint_C \mathbf{B} \cdot d\ell = \oint_S \left[\nabla \times \mathbf{B}(\mathbf{x})\right] \cdot d\mathbf{A}$$

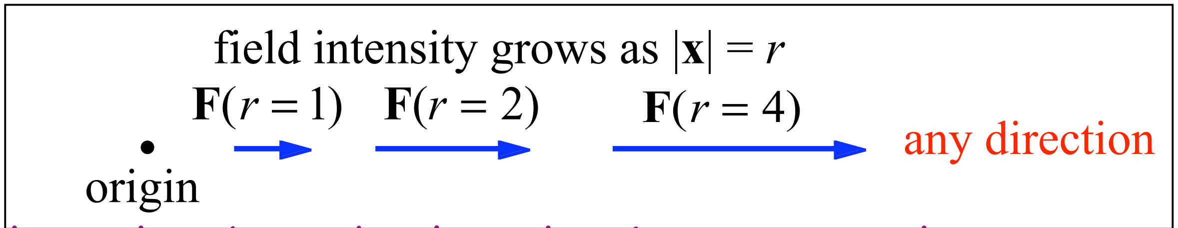
RHR on C gives normal to surface S

Neither side looks particularly simple, however, for constant magnetic fields the right hand side will be shown to be equal to the μ_0•current enclosed.

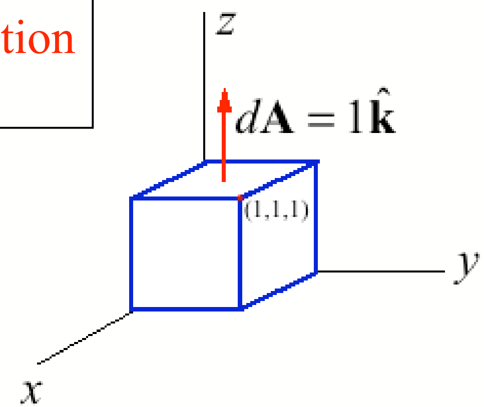
Problem 2.10

- Consider the vector field $\mathbf{F}(\mathbf{x}) = \mathbf{x} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$

  1) Sketch the field



2) Flux through cube, edge length = 1, corner at the orig



- Field in x-z, y-z, or x-y plane is parallel to a cube face therefore, through those 3 faces the flux = 0
- The same flux goes through each of the other 3 faces
- Consider the top face: z = 1, area = 1 .

$$\oint_{Top} \mathbf{F}\cdot d\mathbf{A} = \mathbf{x}\cdot\hat{\mathbf{k}} = z = 1$$

$$\oint_S \mathbf{F}\cdot d\mathbf{A} = 3$$

3) Using Gauss's theorem

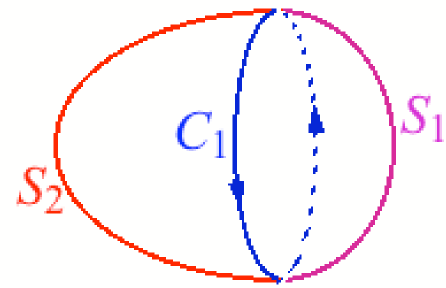
$$\nabla\cdot\mathbf{F} = \frac{\partial F_i}{\partial x_i} = 1+1+1 = 3$$

$$\oint_S \mathbf{F}\cdot d\mathbf{A} = \oint_V \nabla\cdot\mathbf{F}\,dV = 3\oint_V dV = 3$$

Problem 2.11

- Use Gauss's then Stokes's Theorem to prove:

$$\oint_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = -\oint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$$



Gauss's Theorem

Let $S = S_1 + S_2$, and $\mathbf{G} = \nabla \times \mathbf{r}$

$$\oint_S \mathbf{G} \cdot d\mathbf{A} = \oint_V (\nabla \cdot \mathbf{G}) dV$$

See Slide 6 this lecture

$$\oint_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{A} + \oint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = \oint_V \nabla \cdot (\nabla \times \mathbf{F}) dV = 0$$

Stokes's Theorem

$$\oint_{S_1} (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dA = \oint_{C_1} \mathbf{F} \cdot d\ell$$

$$\oint_{S_2} (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dA = \oint_{C_2} \mathbf{F} \cdot d\ell$$

$$\oint_{C_1} = -\oint_{C_2}$$

S_1 outward normal via right hand rule on C_1. S_2 outward normal opposite direction for C_2

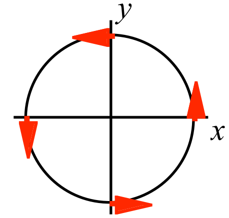
Gauss's & Stokes's Theorems give the same answer

$$\oint_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = -\oint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$$

Vector field Problem 2.9

- Consider the vector field $\mathbf{F}(\mathbf{x}) = \hat{\mathbf{k}} \times \mathbf{x}$

$$\mathbf{F}(\mathbf{x}) = \hat{\mathbf{k}} \times \left(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}\right) = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}} = r\hat{\boldsymbol{\phi}}$$

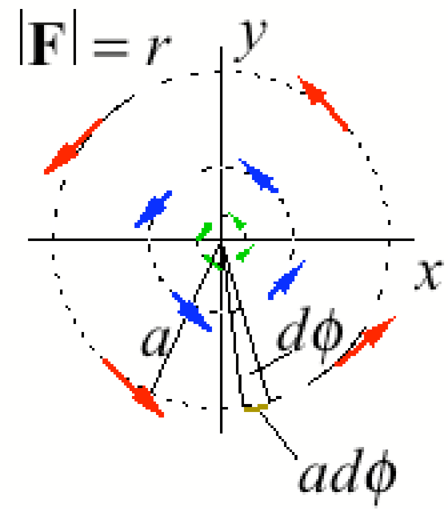


ϕ direction
no z-dependence

b) Line integral over circle radius a

$$\oint_C \mathbf{F} \cdot d\ell = \int_0^{2\pi} \left(a\hat{\boldsymbol{\phi}}\right) \cdot \left(a\,d\phi\,\hat{\boldsymbol{\phi}}\right) = \underline{2\pi a^2}$$

a) Sketch the field



c) Line integral via Stokes's Theorem

$$[\nabla \times \mathbf{F}]_3 = \varepsilon_{3jk} \frac{\partial F_k}{\partial x_j} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 2$$

$$\nabla \times \mathbf{F} = 2\hat{\mathbf{k}}$$

Stokes's Theorem

$$\oint_C \mathbf{F} \cdot d\ell = \oint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{A} \quad = 2\hat{\mathbf{k}} \cdot \hat{\mathbf{k}} \int_0^{2\pi} d\phi \int_0^a r\,dr = \underline{2\pi a^2}$$