
PHY481: Electromagnetism

Vector operators in curvilinear coordinates

Curvilinear coordinates

Cartesian coordinate vector operators

Del

$$\nabla = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}$$

Gradient

$$\nabla V(\mathbf{x}) = \frac{\partial V}{\partial x_i} \hat{\mathbf{e}}_i$$

Divergence

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = \frac{\partial E_i}{\partial x_i}$$

Curl

$$[\nabla \times \mathbf{B}(\mathbf{x})]_i = \varepsilon_{ijk} \frac{\partial B_k}{\partial x_j}$$

Laplacian

$$\nabla^2 V = \frac{\partial^2 V}{\partial x_i^2}$$

Compare coordinate systems - SCALE FACTORS

Coordinates	Unit vectors	Position	Displacement	Scale Factors
u_1, u_2, u_3	$\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$	$\mathbf{x} =$	$d\mathbf{s} =$	h_1, h_2, h_3
Cartesian x, y, z	$\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$	$x \hat{\mathbf{i}} + y \hat{\mathbf{j}} + z \hat{\mathbf{k}}$	$dx \hat{\mathbf{i}} + dy \hat{\mathbf{j}} + dz \hat{\mathbf{k}}$	$1, 1, 1$
Cylindrical r, ϕ, z	$\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{k}}$	$r \hat{\mathbf{r}} + z \hat{\mathbf{k}}$	$dr \hat{\mathbf{r}} + r d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{k}}$	$1, r, 1$
Spherical r, θ, ϕ	$\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$	$r \hat{\mathbf{r}}$	$dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}}$	$1, r, r \sin \theta$

Scale factors

- Scale factors relate displacements to coordinate changes

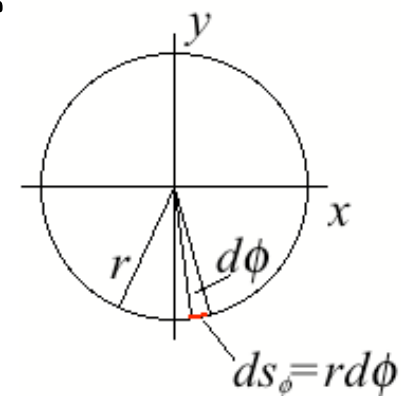
$$d\mathbf{s} = h_1 du_1 \hat{\mathbf{e}}_1 + h_2 du_2 \hat{\mathbf{e}}_2 + h_3 du_3 \hat{\mathbf{e}}_3$$

$$h_i = \frac{ds_i}{du_i}$$

- Example: **cylindrical** coordinate scale factor

$$(u_1, u_2, u_3) = (r, \phi, z)$$

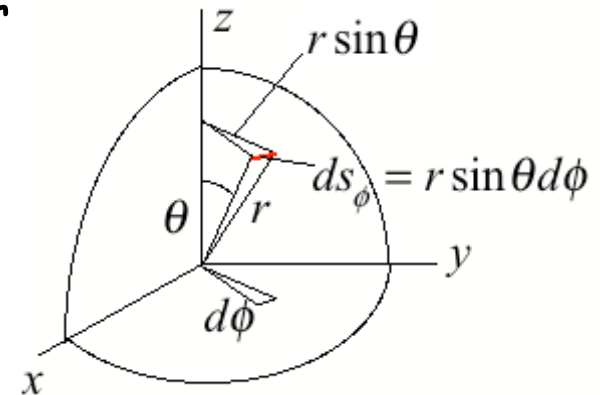
$$ds_\phi = r d\phi; \quad h_\phi = \frac{ds_\phi}{d\phi} = r$$



- Example: **spherical** coordinate scale factor

$$(u_1, u_2, u_3) = (r, \theta, \phi)$$

$$ds_\phi = r \sin \theta d\phi; \quad h_\phi = \frac{ds_\phi}{d\phi} = r \sin \theta$$



Vector operators in curvilinear coordinates

- Operators in terms of coordinates and scale factors

$$\nabla V(\mathbf{x}) = \frac{1}{h_i} \frac{\partial V}{\partial u_i} \hat{\mathbf{e}}_i$$

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 E_1) + \frac{\partial}{\partial u_2} (h_1 h_3 E_2) + \frac{\partial}{\partial u_3} (h_1 h_2 E_3) \right]$$

$$\nabla^2 V(\mathbf{x}) = \frac{1}{h_1 h_2 h_3} \sum_{(ijk)} \frac{\partial}{\partial u_i} \left[\frac{h_j h_k}{h_i} \frac{\partial V}{\partial u_i} \right]$$

(ijk) = cyclic
permutations of (123)

$$[\nabla \times \mathbf{B}(\mathbf{x})]_i = \varepsilon_{ijk} \left[\frac{1}{h_j h_k} \frac{\partial}{\partial u_j} (h_k B_k) \right]$$

Expanding curl in curvilinear coordinates

Cylindrical

$$\hat{\mathbf{r}} \quad [\nabla \times \mathbf{B}(\mathbf{x})]_r = \left[\frac{1}{r} \frac{\partial}{\partial \phi} (B_z) - \frac{\partial}{\partial z} (B_\phi) \right]$$

$$\hat{\phi} \quad [\nabla \times \mathbf{B}(\mathbf{x})]_\phi = \left[\frac{\partial}{\partial z} (B_r) - \frac{\partial}{\partial r} (B_z) \right]$$

$$\hat{\mathbf{k}} \quad [\nabla \times \mathbf{B}(\mathbf{x})]_z = \frac{1}{r} \left[\frac{\partial}{\partial r} (r B_\phi) - \frac{\partial}{\partial \phi} (B_r) \right]$$

Spherical

$$\hat{\mathbf{r}} \quad [\nabla \times \mathbf{B}(\mathbf{x})]_r = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta B_\phi) - \frac{\partial}{\partial \phi} (B_\theta) \right]$$

$$\hat{\theta} \quad [\nabla \times \mathbf{B}(\mathbf{x})]_\theta = \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \phi} (B_r) - \frac{\partial}{\partial r} (r B_\phi) \right]$$

$$\hat{\phi} \quad [\nabla \times \mathbf{B}(\mathbf{x})]_\phi = \frac{1}{r} \left[\frac{\partial}{\partial r} (r B_\theta) - \frac{\partial}{\partial \theta} (B_r) \right]$$

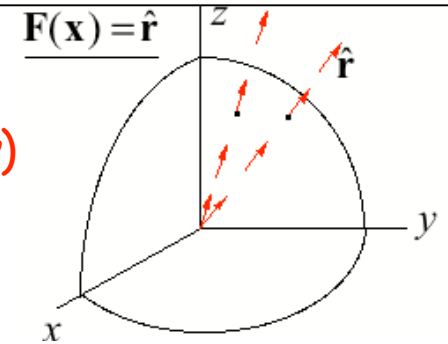
Example 2.7

- In spherical coordinates, calculate divergence and curl & ii

$\mathbf{F}(\mathbf{x}) = 1\hat{\mathbf{r}}$ (constant F in radial direction)

$$\nabla \cdot \hat{\mathbf{r}} = \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} (r^2 \sin \theta F_r) \right] = \frac{2}{r} \quad (\text{field lines spread in } r)$$

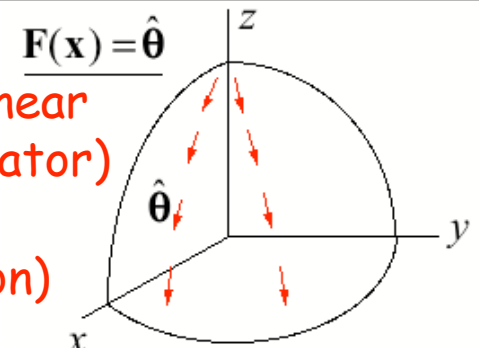
$$\nabla \times \hat{\mathbf{r}} = 0 \quad (\text{as } F_\theta = F_\phi = 0) \quad (\text{field has no "vorticity"})$$



$\mathbf{F}(\mathbf{x}) = 1\hat{\boldsymbol{\theta}}$ (constant F in polar direction)

$$\nabla \cdot \hat{\boldsymbol{\theta}} = \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial \theta} (r \sin \theta F_\theta) \right] = \frac{\cot \theta}{r} \quad (\text{field spreads most near poles, parallel at equator})$$

$$\nabla \times \hat{\boldsymbol{\theta}} = \frac{1}{r} \left[\frac{\partial}{\partial r} (r F_\theta) \right] \hat{\boldsymbol{\phi}} = \frac{1}{r} \hat{\boldsymbol{\phi}} \quad (\text{field curls around } \phi \text{ direction})$$

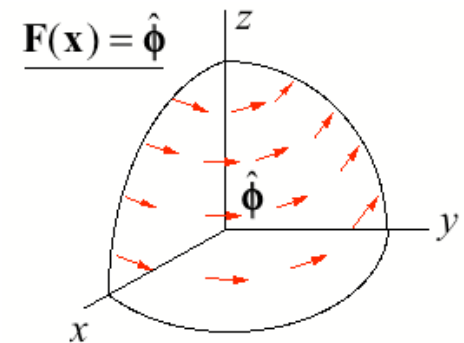


$\mathbf{F}(\mathbf{x}) = 1\hat{\boldsymbol{\phi}}$ (constant F in azimuth direction)

$$\nabla \cdot \hat{\boldsymbol{\phi}} = \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial \phi} (r F_\phi) \right] = 0 \quad (\text{field lines are parallel})$$

$$\nabla \times \hat{\boldsymbol{\phi}} = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta F_\phi) \right] \hat{\mathbf{r}} + \frac{1}{r} \left[-\frac{\partial}{\partial r} (r F_\phi) \right] \hat{\boldsymbol{\theta}} = \frac{1}{r \sin \theta} \hat{\mathbf{k}}$$

$$\hat{\mathbf{k}} = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}} \quad (\text{see problem 2.14}) \quad (\text{field curls around } z \text{ direction})$$



Helmholtz Theorem

- Important identities $\nabla \times (\nabla \psi) = 0$ $\nabla \cdot (\nabla \times \mathbf{A}) = 0$
- Arbitrary function: $\mathbf{H}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) + \mathbf{G}(\mathbf{x})$

"irrotational" part

"solenoidal" part

$$\mathbf{F}(\mathbf{x}) = -\nabla \psi(\mathbf{x}) \quad \mathbf{G}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x})$$

Helmholtz Theorem

Let $\nabla \times \mathbf{H}(\mathbf{x}) = \mathbf{c}(\mathbf{x})$, $\nabla \cdot \mathbf{H}(\mathbf{x}) = d(\mathbf{x})$

and $\mathbf{c}(\mathbf{x})$ & $d(\mathbf{x}) \xrightarrow{r \rightarrow \infty} 0$ faster than r^{-2} and $\mathbf{H}(\mathbf{x}) \xrightarrow{r \rightarrow \infty} 0$

$$\mathbf{H}(\mathbf{x}) = -\nabla \psi + \nabla \times \mathbf{A}$$

$$\psi(\mathbf{x}) = \int \frac{d(\mathbf{x}') d^3 x'}{4\pi |\mathbf{x} - \mathbf{x}'|}$$

$d(\mathbf{x})$ will be charge density/ ϵ_0

$$\mathbf{A}(\mathbf{x}) = \int \frac{\mathbf{c}(\mathbf{x}') d^3 x'}{4\pi |\mathbf{x} - \mathbf{x}'|}$$

$\mathbf{c}(\mathbf{x})$ will be $\mu_0 \cdot$ current density