
PHY481: Electromagnetism

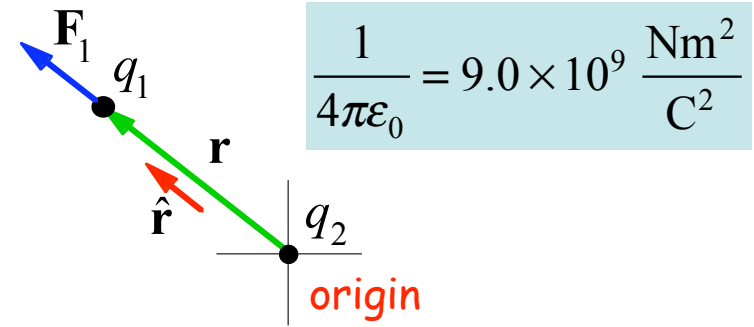
Coulomb's law and the electric field

Coulomb's Law

Force between two charges

Lectures 1 - 2

$$\mathbf{F}_1 = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{\mathbf{r}}$$



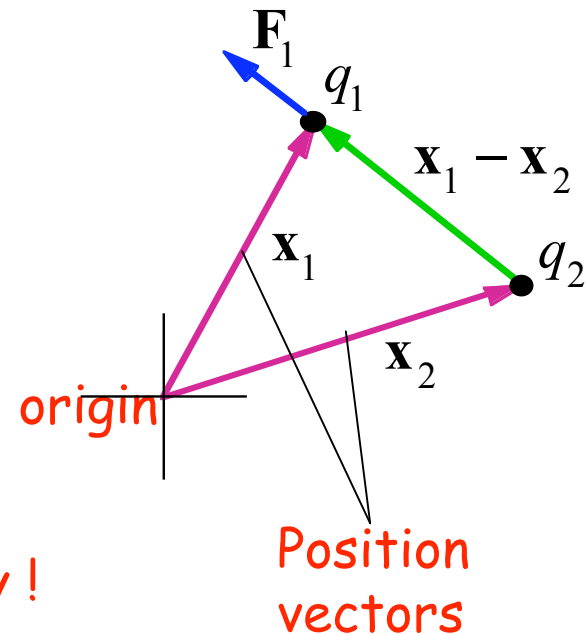
From now on,

$$\mathbf{F}_1 = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3}$$

Not a typo !

Unit vector with a direction from 2 to 1

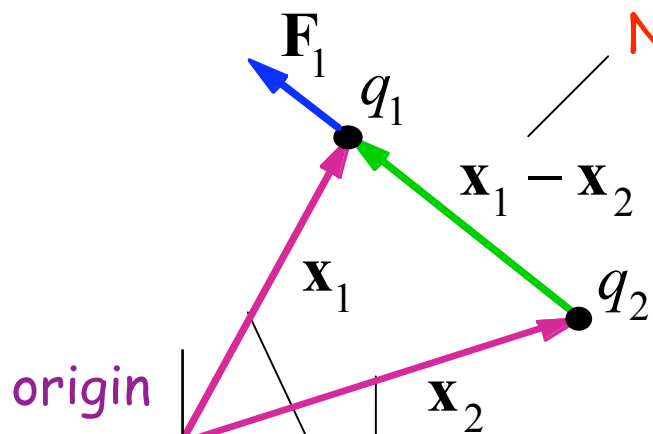
$$\frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|}$$



Erase previous version from your memory !

Why change notation?

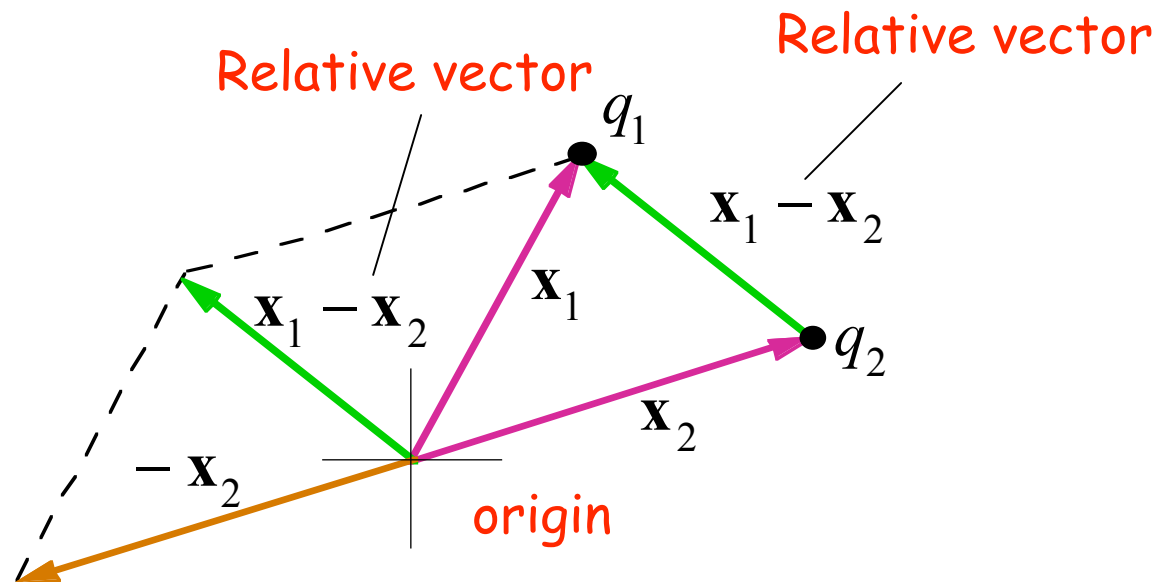
- Position vectors w.r.t. a fixed origin.



Not a position vector

- Relative vectors are origin independent

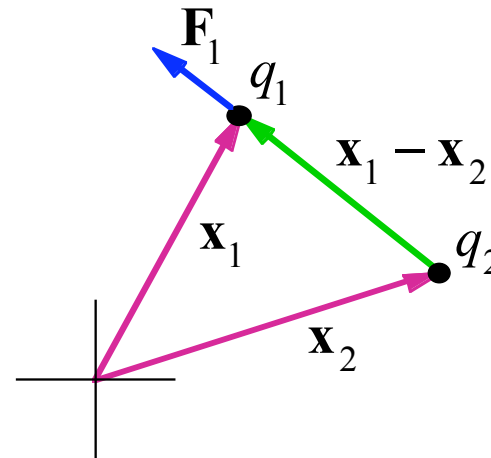
Construct $\mathbf{x}_1 - \mathbf{x}_2$



Multiple charges

- Coulomb's law (2 charges)

$$\mathbf{F}_1 = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3}$$

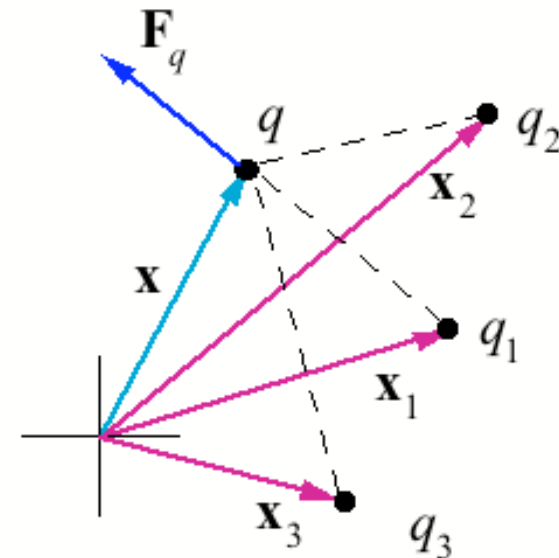


- Force on charge q from N charges, $q_1 \cdots q_N$

$$\mathbf{F}_q(\mathbf{x}) = \sum_{k=1}^N \frac{q q_k}{4\pi\epsilon_0} \frac{\mathbf{x} - \mathbf{x}_k}{|\mathbf{x} - \mathbf{x}_k|^3}$$

- **Reserve \mathbf{r}** for the position vector in spherical coordinates

$$\mathbf{x} = \mathbf{r} = r \hat{\mathbf{r}}$$



Electric field

- Coulomb's law for discrete charges q , and q_k , $k=1,...,N$

$$\mathbf{F}_q(\mathbf{x}) = \sum_{k=1}^N \frac{qq_k}{4\pi\epsilon_0} \frac{\mathbf{x} - \mathbf{x}_k}{|\mathbf{x} - \mathbf{x}_k|^3}$$

- Use a very small charge q at the point \mathbf{x} to measure the force vector \mathbf{F}_q . The electric field \mathbf{E} at the point \mathbf{x} is the limit

$$\mathbf{E}(\mathbf{x}) = \lim_{q \rightarrow 0} \frac{\mathbf{F}_q}{q}$$

- For discrete charges q , and q_k , $k=1,...,N$ the field is

$$\mathbf{E}(\mathbf{x}) = \sum_{k=1}^N \frac{q_k}{4\pi\epsilon_0} \frac{\mathbf{x} - \mathbf{x}_k}{|\mathbf{x} - \mathbf{x}_k|^3}$$

- At the point \mathbf{x} , an electric field $\mathbf{E}(\mathbf{x})$ exerts a force \mathbf{F} on a charge q such that

$$\mathbf{F}_q = q\mathbf{E}(\mathbf{x})$$

Charge motion in an electric field

In an electric field E_0 in the z direction, a proton travels with an initial velocity v_0 in the x direction. What is its trajectory?

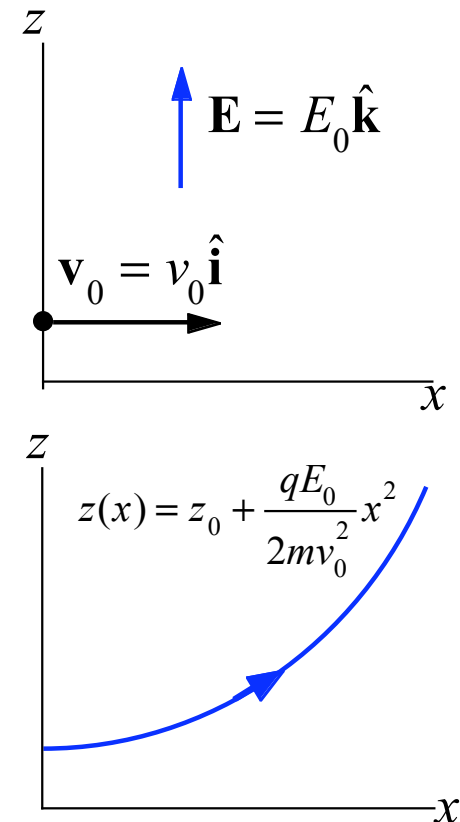
Acceleration: $\mathbf{a} = \frac{\mathbf{F}}{m} = \frac{q\mathbf{E}}{m} = \frac{qE_0}{m} \hat{\mathbf{k}}$

Displacement: $\mathbf{s}(t) = \mathbf{v}_0 t + \frac{1}{2} \mathbf{a} t^2$
 $= (v_0 t) \hat{\mathbf{i}} + \frac{qE_0 t^2}{2m} \hat{\mathbf{k}}$

Position: $x = x_0 + v_0 t; \quad z = z_0 + \frac{qE_0}{2m} t^2$

Trajectory is coupled:

$$z(x) = z_0 + \frac{qE_0}{2mv_0^2} (x - x_0)^2 \quad (\text{Parabola})$$



Electric field calculation

Position vectors

$$\mathbf{x} = a(\hat{\mathbf{j}} + \hat{\mathbf{k}}) \quad \mathbf{x}_1 = a\hat{\mathbf{k}}, \mathbf{x}_2 = 0, \mathbf{x}_3 = a\hat{\mathbf{j}}$$

Relative vectors

$$\mathbf{x} - \mathbf{x}_1 = a(\hat{\mathbf{j}} + \hat{\mathbf{k}}) - a\hat{\mathbf{k}} = a\hat{\mathbf{j}}$$

$$\mathbf{x} - \mathbf{x}_2 = a(\hat{\mathbf{j}} + \hat{\mathbf{k}}) - 0 = a(\hat{\mathbf{j}} + \hat{\mathbf{k}})$$

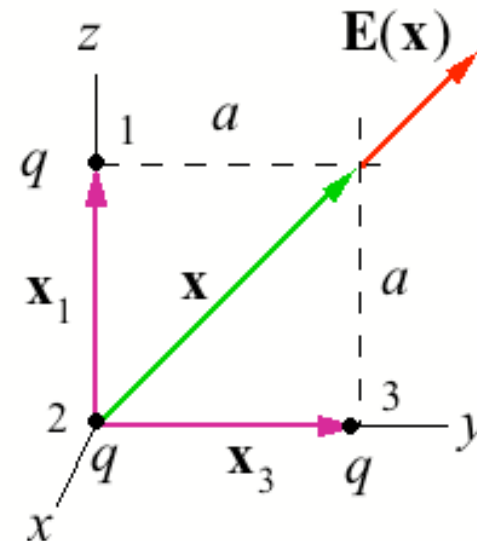
$$\mathbf{x} - \mathbf{x}_3 = a(\hat{\mathbf{j}} + \hat{\mathbf{k}}) - a\hat{\mathbf{j}} = a\hat{\mathbf{k}}$$

Electric field

$$\begin{aligned} \mathbf{E}(\mathbf{x}) &= \frac{q}{4\pi\epsilon_0} \sum_{k=1}^N \frac{\mathbf{x} - \mathbf{x}_k}{|\mathbf{x} - \mathbf{x}_k|^3} \\ &= \frac{q}{4\pi\epsilon_0} \left[\frac{a\hat{\mathbf{j}}}{a^3} + \frac{a(\hat{\mathbf{j}} + \hat{\mathbf{k}})}{2\sqrt{2}a^3} + \frac{a\hat{\mathbf{k}}}{a^3} \right] = \frac{q}{4\pi\epsilon_0 a^2} \left(1 + \frac{\sqrt{2}}{2} \right) (\hat{\mathbf{j}} + \hat{\mathbf{k}}) \end{aligned}$$

Charge q at 3 corners of square side a .

Find Electric field at 4th corner

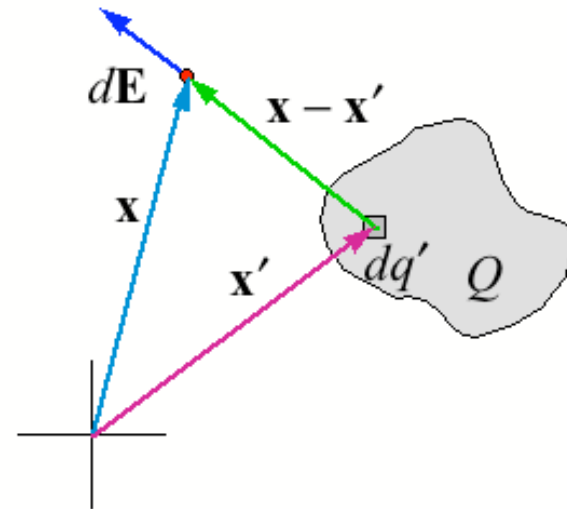


Continuous charge distributions

Distribution:	<u>Volume</u>	<u>Surface</u>	<u>Line</u>
Charge density:	$\rho(\mathbf{x}')$	$\sigma(\mathbf{x}')$	$\lambda(\mathbf{x}')$
Total Charge:	$Q = \int \rho(\mathbf{x}') d^3 x'$	$Q = \int \sigma(\mathbf{x}') d^2 x'$	$Q = \int \lambda(\mathbf{x}') dx'$

- Integration over charge distribution

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \rho(\mathbf{x}') d^3 x'$$



Linear charge distributions

What is the electric field at a distance h above the bisector of a line length 2ℓ , with charge density λ .

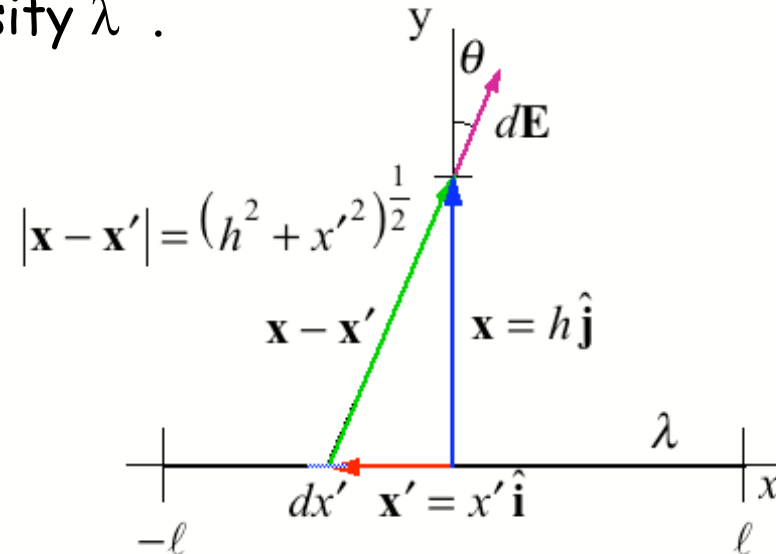
Direct substitution

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \rho(\mathbf{x}') d^3x'$$

$$\mathbf{E}(\mathbf{x}) = \frac{\lambda}{4\pi\epsilon_0} \int (h\hat{\mathbf{j}} - x'\hat{\mathbf{i}})(h^2 + x'^2)^{-\frac{3}{2}} dx'$$

$$= \frac{\lambda h\hat{\mathbf{j}}}{4\pi\epsilon_0} \left[\int_{-\ell}^{\ell} (h^2 + x'^2)^{-\frac{3}{2}} dx' \right] - \frac{\lambda\hat{\mathbf{i}}}{4\pi\epsilon_0} \left[\int_{-\ell}^{\ell} x'(h^2 + x'^2)^{-\frac{3}{2}} dx' \right]$$

$$E_y(h) = \frac{\lambda}{4\pi\epsilon_0 h} \left[x'(h^2 + x'^2)^{-\frac{1}{2}} \right]_{-\ell}^{+\ell} = \frac{\lambda\ell}{2\pi\epsilon_0 h(h^2 + \ell^2)^{1/2}}$$



Odd Integrand

Surface charge distribution

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \rho(\mathbf{x}') d^3x' \quad |\mathbf{x} - \mathbf{x}'| = (H^2 + x'^2 + y'^2)^{\frac{1}{2}}$$

$$= \frac{1}{4\pi\epsilon_0} \int \frac{H\hat{\mathbf{k}} - y'\hat{\mathbf{j}} - x'\hat{\mathbf{i}}}{(H^2 + x'^2 + y'^2)^{\frac{3}{2}}} \sigma dx' dy'$$

$-y'\hat{\mathbf{j}} - x'\hat{\mathbf{i}}$ terms are odd

$$E_z = \frac{\sigma H}{4\pi\epsilon_0} \int_{-\ell}^{\ell} dy' \int_{-\ell}^{\ell} dx' ([H^2 + y'^2] + x'^2)^{-\frac{3}{2}}$$

$$= \frac{\sigma H \ell_x}{2\pi\epsilon_0} \int_{-\ell_y}^{\ell_y} dy' \left[\frac{1}{[H^2 + y'^2]([H^2 + \ell_x^2] + y'^2)^{\frac{1}{2}}} \right]$$

$$E_z = \frac{\sigma}{\pi\epsilon_0} \arctan \left[\frac{\ell_x \ell_y}{H(H^2 + \ell_x^2 + \ell_y^2)^{\frac{1}{2}}} \right]$$

