
PHY481: Electromagnetism

HW2 and Discussion

Problem 2.2

Parllelogram Area

$$|\mathbf{A} \times \mathbf{B}| = |A_y| |B_x| = |\mathbf{A}| |\mathbf{B}| \sin \theta$$

Parlleleped Volume

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \sin \theta \hat{\mathbf{k}}$$

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} &= (|\mathbf{A}| |\mathbf{B}| \sin \theta) \hat{\mathbf{k}} \cdot \mathbf{C} \\ &= |\mathbf{A}| |\mathbf{B}| |\mathbf{C}| \sin \theta \cos \phi \\ &= \text{Volume} \end{aligned}$$

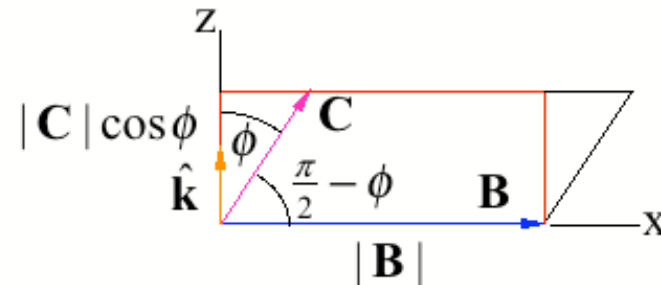
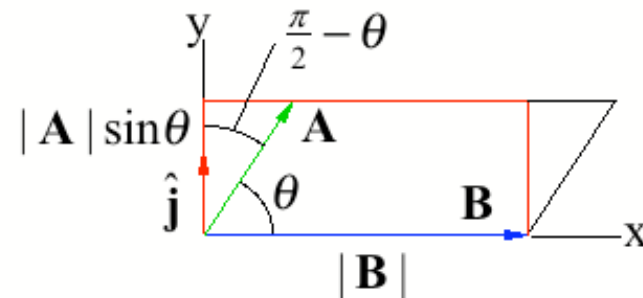
Show $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \text{Volume}$

$$\begin{aligned} \mathbf{B} \times \mathbf{C} &= \hat{\mathbf{j}} |\mathbf{B}| |\mathbf{C}| \sin\left(\frac{\pi}{2} - \phi\right) \\ &= \hat{\mathbf{j}} |\mathbf{B}| |\mathbf{C}| \cos \phi \end{aligned}$$

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \mathbf{A} \cdot \hat{\mathbf{j}} |\mathbf{B}| |\mathbf{C}| \cos \phi \\ &= |\mathbf{A}| |\mathbf{B}| |\mathbf{C}| \cos \phi \cos\left(\frac{\pi}{2} - \theta\right) \\ &= |\mathbf{A}| |\mathbf{B}| |\mathbf{C}| \cos \phi \sin \theta \\ &= \text{Volume} \end{aligned}$$

Area of rectangle

$$\text{Area} = |\mathbf{A}| |\mathbf{B}| \sin \theta$$



Volume of rectangular solid

$$\text{Volume} = |\mathbf{A}| \sin \theta |\mathbf{B}| |\mathbf{C}| \cos \phi$$

Problem 2.3

Tensor product:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Cross product

components i & l:

$$(\mathbf{A} \times \mathbf{B})_i = \epsilon_{ijk} A_j B_k; \quad (\mathbf{C} \times \mathbf{D})_l = \epsilon_{lmn} C_m D_n$$

Dot product:

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= (\epsilon_{ijk} A_j B_k) (\epsilon_{lmn} C_m D_n) \\ &= \epsilon_{jki} \epsilon_{lmn} A_j B_k C_m D_n \end{aligned}$$

Replace tensor product:

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) A_j B_k C_m D_n$$

Replace l,m,n with j or k:

$$\begin{aligned} &= A_j B_k C_j D_k - A_j B_k C_k D_j \\ &= (A_j C_j)(B_k D_k) - (A_j D_j)(B_k C_k) \end{aligned}$$

Identify dot products:

$$\underline{(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D})} = \underline{(\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})}$$

Problem 2.6

$$\underline{\nabla \cdot (g\mathbf{F})} = \frac{\partial}{\partial x_i} (gF_i) = F_i \frac{\partial g}{\partial x_i} + g \frac{\partial F_i}{\partial x_i} = \underline{\mathbf{F} \cdot \nabla g + g \nabla \cdot \mathbf{F}} \text{ Scalar!}$$

$$[\nabla \times (g\mathbf{F})]_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} (gF_k) = \varepsilon_{ijk} \frac{\partial g}{\partial x_j} F_k + g \varepsilon_{ijk} \frac{\partial F_k}{\partial x_j}$$

$$\underline{\nabla \times (g\mathbf{F}) = (\nabla g) \times \mathbf{F} + g \nabla \times \mathbf{F}}$$

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \frac{\partial}{\partial x_i} (\varepsilon_{ijk} F_j G_k) = \varepsilon_{ijk} \left(\frac{\partial F_j}{\partial x_i} \right) G_k + \varepsilon_{ijk} F_j \frac{\partial G_k}{\partial x_i} \quad \text{Even vs Odd permutations}$$

$$\underline{\nabla \cdot (\mathbf{F} \times \mathbf{G}) = G_k \varepsilon_{kij} \left(\frac{\partial F_j}{\partial x_i} \right) - F_j \varepsilon_{jik} \frac{\partial G_k}{\partial x_i} = \underline{\mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})}}$$

$\varepsilon_{ijk} = \varepsilon_{kij}$, and $\varepsilon_{ijk} = -\varepsilon_{jik}$

Like $\mathbf{A} \times \mathbf{B} \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

$$[\nabla \times (\nabla \times \mathbf{F})]_i = [\nabla \times \mathbf{G}]_i = \varepsilon_{ijk} \frac{\partial G_k}{\partial x_j}$$

$$G_k = (\nabla \times \mathbf{F})_k = \varepsilon_{klm} \frac{\partial F_m}{\partial x_l}$$

$$[\nabla \times (\nabla \times \mathbf{F})]_i = \varepsilon_{ijk} \varepsilon_{klm} \frac{\partial^2 F_m}{\partial x_j \partial x_l} = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \frac{\partial^2 F_m}{\partial x_j \partial x_l}$$

$$\nabla \times (\nabla \times \mathbf{F}) = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} \left(\frac{\partial F_j}{\partial x_j} \right) - \frac{\partial^2}{\partial x_j \partial x_j} (F_i \hat{\mathbf{e}}_i)$$

$$\underline{\nabla \times (\nabla \times \mathbf{F}) = \underline{\nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}}}$$

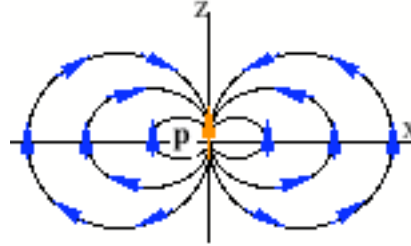
Problem 2.8

$$\nabla\Phi = \nabla\left(\frac{\mathbf{p} \cdot \mathbf{x}}{r^3}\right) = \frac{\nabla(\mathbf{p} \cdot \mathbf{x})}{r^3} + (\mathbf{p} \cdot \hat{\mathbf{r}}) \nabla\left(\frac{1}{r^2}\right)$$

$$\nabla(\mathbf{p} \cdot \mathbf{x}) = \hat{\mathbf{e}}_j \frac{\partial}{\partial x_j} p_i x_i$$

$$= \hat{\mathbf{e}}_j p_i \frac{\partial x_i}{\partial x_j} = p_i \hat{\mathbf{e}}_i = \mathbf{p}$$

$$\nabla\Phi = \frac{1}{r^3}(\mathbf{p} - 3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}})$$



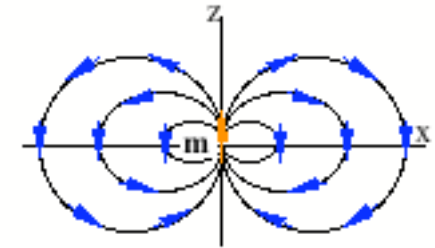
$$\nabla\left(\frac{1}{r^3}\right) = -3\frac{\hat{\mathbf{r}}}{r^4}$$

$$\nabla \times \mathbf{A} = \nabla \times \left(\frac{\mathbf{m} \times \mathbf{x}}{r^3}\right) = \frac{1}{r^3} \nabla \times (\mathbf{m} \times \mathbf{x}) + \nabla\left(\frac{1}{r^3}\right) \times (\mathbf{m} \times \mathbf{x})$$

$$[\nabla \times (\mathbf{m} \times \mathbf{x})]_i = \varepsilon_{ijk} \varepsilon_{klm} m_l \frac{\partial x_m}{\partial x_j}$$

$$= \varepsilon_{ijk} \varepsilon_{klj} m_l = (\delta_{il} \delta_{jj} - \delta_{ij} \delta_{jl}) m_l$$

$$= \delta_{jj} m_i - m_i = 3m_i - m_i = 2m_i = 2\mathbf{m}$$



$$\begin{aligned} \nabla \times \mathbf{A} &= \frac{2\mathbf{m}}{r^3} - 3\frac{\hat{\mathbf{r}} \times (\mathbf{m} \times \hat{\mathbf{r}})}{r^3} = \frac{1}{r^3} [2\mathbf{m} - 3\{\mathbf{m}(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}) - \hat{\mathbf{r}}(\mathbf{m} \cdot \hat{\mathbf{r}})\}] \\ &= \frac{1}{r^3} [-\mathbf{m} + 3\hat{\mathbf{r}}(\mathbf{m} \cdot \hat{\mathbf{r}})] \end{aligned}$$

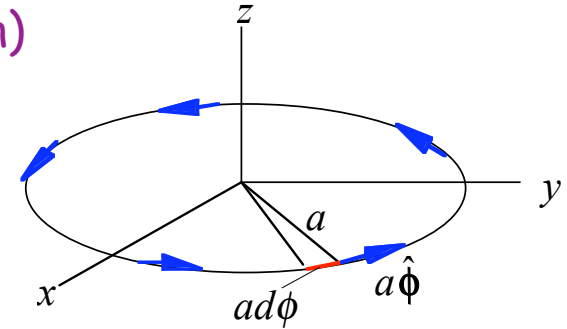
Problem 2.9

$$\mathbf{F}(\mathbf{x}) = \hat{\mathbf{k}} \times \mathbf{x} = \hat{\mathbf{k}} \times (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}} \quad (\text{Cartesian})$$

$$\mathbf{F}(\mathbf{x}) = \hat{\mathbf{k}} \times \mathbf{r} = r\hat{\mathbf{k}} \times \hat{\mathbf{r}} = r\hat{\phi} \quad (\text{Cylindrical})$$

$$d\ell = \hat{\phi} a d\phi$$

$$\oint_C \mathbf{F} \cdot d\ell = \int_0^{2\pi} a\hat{\phi} \cdot \hat{\phi} a d\phi = \underline{2\pi a^2}$$



Stokes's theorem

$$\oint_C \mathbf{F} \cdot d\ell = \oint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$$

$$\mathbf{F}(\mathbf{x}) = r\hat{\phi} \quad \text{Only phi component} \quad F_\phi = r$$

$$\nabla \times \mathbf{F} = \hat{\mathbf{r}} \left(-\frac{\partial F_\phi}{\partial z} \right) + \hat{\mathbf{k}} \left(\frac{1}{r} \frac{\partial}{\partial r} (r F_\phi) \right) = 2\hat{\mathbf{k}}$$

$$d\mathbf{A} = \hat{\mathbf{k}} r dr d\phi$$

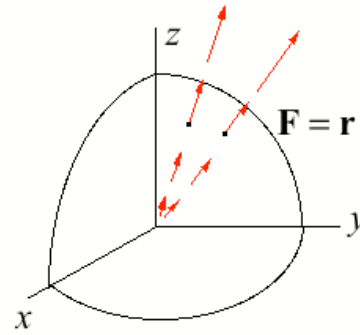
$$\oint_C \mathbf{F} \cdot d\ell = \oint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = \oint_S 2\hat{\mathbf{k}} \cdot \hat{\mathbf{k}} r dr d\phi$$

$$= 2 \int_0^a r dr \int_0^{2\pi} d\phi = \underline{2\pi a^2}$$

Problem 2.10

Position vector in 2 systems

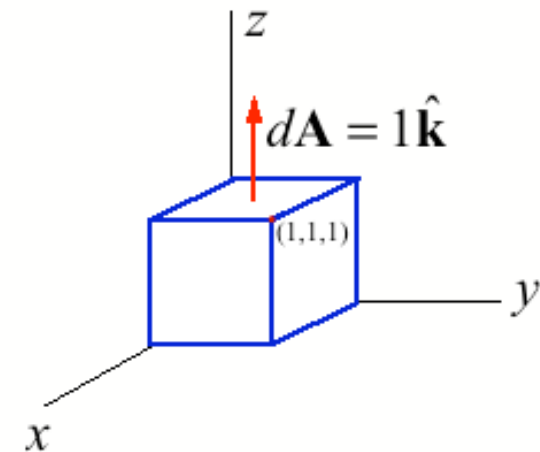
$$\mathbf{F}(\mathbf{x}) = \mathbf{x} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = r\hat{\mathbf{r}}$$



Back, left, and bottom faces of cube are parallel to field

Flux through the top face: $\oint_{Top} \mathbf{F} \cdot d\mathbf{A} = \mathbf{x} \cdot \hat{\mathbf{k}} = z = 1$

Total flux: $\oint_S \mathbf{F} \cdot d\mathbf{A} = \underline{3}$



Gauss's theorem: $\oint_S \mathbf{F} \cdot d\mathbf{A} = \oint_V \nabla \cdot \mathbf{F} dV$

Divergence of F

$$\nabla \cdot \mathbf{F} = \frac{\partial F_i}{\partial x_i} = 1 + 1 + 1 = 3$$

Volume integral

$$\oint_V \nabla \cdot \mathbf{F} dV = 3 \oint_V dV = \underline{3}$$

Problem 2.12

$$\underline{f = x^2 + y^2 + z^2} \quad \nabla f = \hat{\mathbf{e}}_i \frac{\partial f}{\partial x_i} = 2(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) = 2\mathbf{x} = \underline{2r\hat{\mathbf{r}}}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_i \partial x_i} = \underline{6}$$

$$\underline{f = r^2} \quad \nabla f = \hat{\mathbf{r}} \frac{\partial f}{\partial r} = \underline{2r\hat{\mathbf{r}}}$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} r^2 \right) = \underline{6}$$

$$\underline{f = (x^2 + y^2 + z^2)^{1/2}} \quad \nabla f = \hat{\mathbf{e}}_i \frac{\partial f}{\partial x_i} = \frac{(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})}{(x^2 + y^2 + z^2)^{1/2}} = \frac{\mathbf{r}}{r} = \underline{\hat{\mathbf{r}}}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_i \partial x_i} = \frac{2}{(x^2 + y^2 + z^2)^{1/2}} = \underline{\frac{2}{r}}$$

$$\underline{f = r}$$

$$\nabla f = \frac{\partial}{\partial r}(r)\hat{\mathbf{r}} = \underline{\hat{\mathbf{r}}}$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} r \right) = \underline{\frac{2}{r}}$$

$$\underline{f = (x^2 + y^2 + z^2)^{-1/2}} \quad \nabla f = \hat{\mathbf{e}}_i \frac{\partial f}{\partial x_i} = -\frac{(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})}{(x^2 + y^2 + z^2)^{3/2}} = -\frac{\hat{\mathbf{r}}}{r^2}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_i \partial x_i} = (3-3) \frac{(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}} = \underline{0}$$

$$\underline{f = r^{-1}}$$

$$\nabla f = \frac{\partial}{\partial r}(r^{-1})\hat{\mathbf{r}} = -\frac{\hat{\mathbf{r}}}{r^2}$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} r^{-1} \right) = \underline{0}$$

Problem 2.13

$$\underline{\mathbf{x}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

$$\nabla \cdot (\underline{\mathbf{x}}) = \frac{\partial}{\partial x_i} (x_i) = \underline{3}$$

$$\nabla \times (\underline{\mathbf{x}}) = \epsilon_{ijk} \frac{\partial}{\partial x_j} (x_k) = \epsilon_{ikk} = \underline{0}$$

$$\underline{\mathbf{x}} = r\hat{\mathbf{r}}$$

$$\nabla \cdot (r\hat{\mathbf{r}}) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^3) = \underline{3}$$

$$\nabla \times (r\hat{\mathbf{r}}) = \underline{0}$$

$$\underline{\mathbf{x}} / r = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{(x^2 + y^2 + z^2)^{1/2}}$$

$$\nabla \cdot (\underline{\mathbf{x}} / r) = \frac{\partial}{\partial x_i} \left[\frac{x_i}{r} \right] = \underline{\frac{2}{r}}$$

$$[\nabla \times (\underline{\mathbf{x}} / r)]_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} \left[\frac{x_k}{r} \right] = -\epsilon_{ijk} \frac{x_k x_j}{r^3} = \underline{0}$$

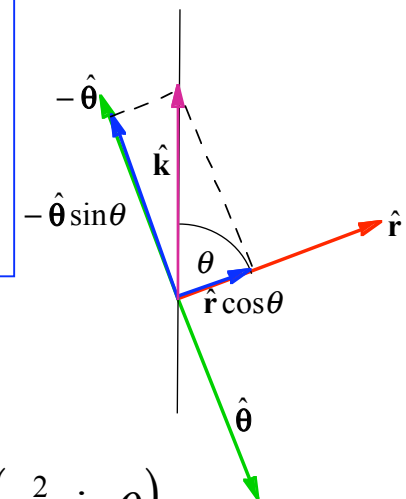
$$\underline{\mathbf{x}} / r = \hat{\mathbf{r}}$$

$$\nabla \cdot \hat{\mathbf{r}} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2) = \underline{\frac{2}{r}}$$

$$\nabla \times \hat{\mathbf{r}} = \underline{0}$$

|unit vectors|=1

$\hat{\mathbf{k}}$ construction



$$\hat{\mathbf{k}} \times \underline{\mathbf{x}} = \hat{\mathbf{k}} \times (x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) = x\hat{\mathbf{j}} - y\hat{\mathbf{i}}$$

$$\nabla \cdot (x\hat{\mathbf{j}} - y\hat{\mathbf{i}}) = \underline{0}$$

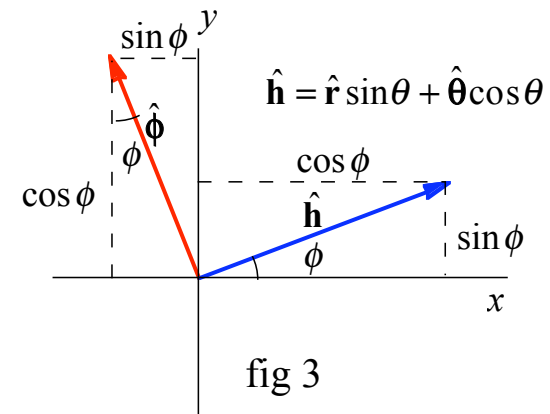
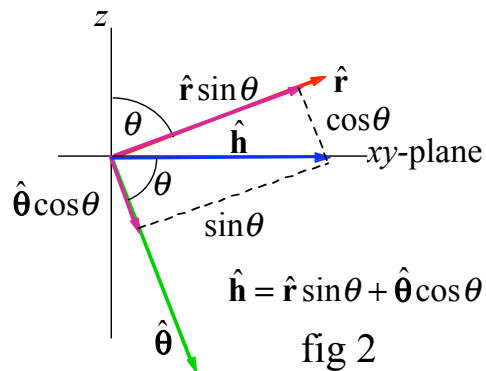
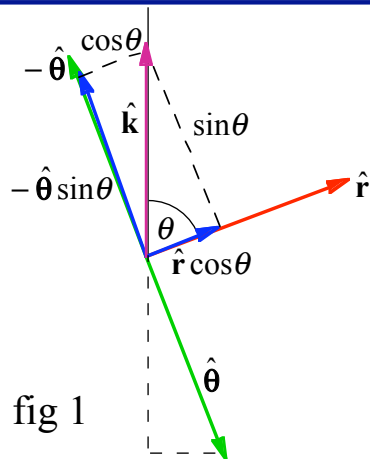
$$\nabla \times (x\hat{\mathbf{j}} - y\hat{\mathbf{i}}) = \underline{2\hat{\mathbf{k}}}$$

$$\hat{\mathbf{k}} \times \underline{\mathbf{x}} = r\hat{\mathbf{k}} \times \hat{\mathbf{r}} = r \sin \theta \hat{\boldsymbol{\phi}}$$

$$\nabla \cdot (r \sin \theta \hat{\boldsymbol{\phi}}) = \underline{0}$$

$$\begin{aligned} \nabla \times (r \sin \theta \hat{\boldsymbol{\phi}}) &= \frac{\hat{\mathbf{r}}}{r \sin \theta} \frac{\partial}{\partial \theta} (r \sin^2 \theta) - \frac{\hat{\boldsymbol{\theta}}}{r} \frac{\partial}{\partial r} (r^2 \sin \theta) \\ &= 2(\hat{\mathbf{r}} \cos \theta - \hat{\boldsymbol{\theta}} \sin \theta) = \underline{2\hat{\mathbf{k}}} \end{aligned}$$

Problem 2.14



see fig. 1: $\hat{\mathbf{k}} = \hat{\mathbf{r}} \cos \theta - \hat{\boldsymbol{\theta}} \sin \theta$

see figs 2&3: $\hat{\mathbf{j}} = \hat{\mathbf{r}} \sin \theta \sin \phi + \hat{\boldsymbol{\theta}} \cos \theta \sin \phi + \hat{\boldsymbol{\phi}} \cos \phi$

$\hat{\mathbf{i}} = \hat{\mathbf{r}} \sin \theta \cos \phi + \hat{\boldsymbol{\theta}} \cos \theta \cos \phi - \hat{\boldsymbol{\phi}} \sin \phi$

$$\mathbf{R} = \begin{pmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{pmatrix}$$

$$\mathbf{R}^T = \begin{pmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{pmatrix}$$

$$\hat{\mathbf{r}} = \hat{\mathbf{i}} \sin \theta \cos \phi + \hat{\mathbf{j}} \sin \theta \sin \phi + \hat{\mathbf{k}} \cos \theta$$

$$\hat{\boldsymbol{\theta}} = \hat{\mathbf{i}} \cos \theta \cos \phi + \hat{\mathbf{j}} \cos \theta \sin \phi - \hat{\mathbf{k}} \sin \theta$$

$$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{i}} \sin \phi + \hat{\mathbf{j}} \cos \phi$$

Problem 2.19

$$\mathbf{F}(\mathbf{x}) = C\mathbf{x} = C(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \quad \nabla \cdot \mathbf{F} = \frac{\partial F_i}{\partial x_i} = 3C$$

Gauss's theorem: $\oint_S \mathbf{F} \cdot d\mathbf{A} = \oint_V (\nabla \cdot \mathbf{F}) dV = 3C(V) = \underline{3C\ell^3}$

Direct calculation: $\oint_S \mathbf{F} \cdot d\mathbf{A} = 6 \oint_S \mathbf{x} \cdot \hat{\mathbf{k}} dA = 6Cz \int_{-\ell/2}^{\ell/2} dx \int_{-\ell/2}^{\ell/2} dy = 6C \frac{\ell}{2} \ell^2 = \underline{3C\ell^3}$

Problem 2.22

Given: $h(\mathbf{x}) = (\mathbf{x} \times \mathbf{A}) \cdot (\mathbf{x} \times \mathbf{B})$ **Prove:** $\nabla h(\mathbf{x}) = \mathbf{A} \times (\mathbf{x} \times \mathbf{B}) + \mathbf{B} \times (\mathbf{x} \times \mathbf{A})$

Cross products: $(\mathbf{x} \times \mathbf{A})_i = \epsilon_{ijk} x_j A_k$ $(\mathbf{x} \times \mathbf{B})_i = \epsilon_{ilm} x_l A_m$

Collect terms: $h(\mathbf{x}) = (\mathbf{x} \times \mathbf{A})_i (\mathbf{x} \times \mathbf{B})_i = \epsilon_{ijk} x_j A_k \epsilon_{ilm} x_l A_m$

$$\begin{aligned} \text{Gradient: } [\nabla h(\mathbf{x})]_n &= \frac{\partial}{\partial x_n} [\epsilon_{ijk} \epsilon_{ilm} x_j A_k x_l B_m] \\ &= \frac{\partial}{\partial x_n} [(\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) x_j A_k x_l B_m] \\ &= \frac{\partial}{\partial x_n} [x_j x_j A_k B_k - x_k A_k x_j B_j] \\ &= 2x_n A_k B_k - A_n x_j B_j - x_k A_k B_n \end{aligned}$$

$$\begin{aligned} \text{Group terms: } [\nabla h(\mathbf{x})]_n &= (x_n A_k B_k - x_k A_k B_n) + (x_n A_k B_k - A_n x_j B_j) \\ \nabla h(\mathbf{x}) &= [\mathbf{x}(\mathbf{A} \cdot \mathbf{B}) - \mathbf{B}(\mathbf{A} \cdot \mathbf{x})] + [\mathbf{x}(\mathbf{B} \cdot \mathbf{A}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{x})] \end{aligned}$$

Triple cross product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

$$\mathbf{A} \times (\mathbf{x} \times \mathbf{B}) = \mathbf{x}(\mathbf{A} \cdot \mathbf{B}) - \mathbf{B}(\mathbf{A} \cdot \mathbf{x})$$

$$\mathbf{B} \times (\mathbf{x} \times \mathbf{A}) = \mathbf{x}(\mathbf{B} \cdot \mathbf{A}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{x})$$

$$\underline{\underline{\nabla h(\mathbf{x}) = [\mathbf{A} \times (\mathbf{x} \times \mathbf{B})] + [\mathbf{B} \times (\mathbf{x} \times \mathbf{A})]}}$$

Problem 2.24

Line integral: $\oint_C \mathbf{F} \cdot d\ell = r \int_0^{2\pi} d\phi = \underline{2\pi r}$

$$\mathbf{F} = \hat{\phi}$$

Stokes's theorem: $\oint_C \mathbf{F} \cdot d\ell = \oint_H (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$

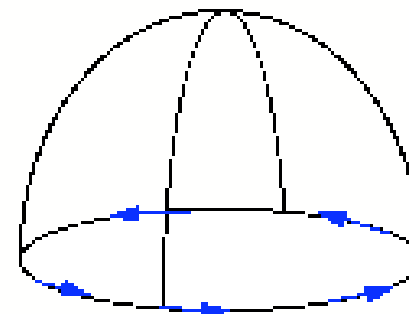
Curl of F:
$$\begin{aligned} \nabla \times \mathbf{F} &= \nabla \times \hat{\phi} = \frac{\hat{\mathbf{r}}}{r \sin \theta} \left(\frac{\partial}{\partial \theta} \sin \theta \right) + \frac{\hat{\boldsymbol{\theta}}}{r} \left(-\frac{\partial}{\partial r} r \right) \\ &= \frac{\hat{\mathbf{r}} \cos \theta}{r \sin \theta} - \frac{\hat{\boldsymbol{\theta}}}{r} = \frac{1}{r \sin \theta} (\hat{\mathbf{r}} \cos \theta - \hat{\boldsymbol{\theta}} \sin \theta) \end{aligned}$$

Integrand: $d\mathbf{A} = \hat{\mathbf{r}} r^2 \sin \theta d\theta d\phi$

$$\begin{aligned} (\nabla \times \mathbf{F}) \cdot d\mathbf{A} &= \frac{1}{r \sin \theta} (\hat{\mathbf{r}} \cos \theta - \hat{\boldsymbol{\theta}} \sin \theta) \cdot (\hat{\mathbf{r}} r^2 \sin \theta d\theta d\phi) \\ &= r \cos \theta d\theta d\phi \end{aligned}$$

Hemisphere Surface integral

$$\begin{aligned} \oint_H (\nabla \times \mathbf{F}) \cdot d\mathbf{A} &= r \int_0^{\pi/2} \cos \theta d\theta \int_0^{2\pi} d\phi \\ &= 2\pi r \sin \theta \Big|_0^{\pi/2} = \underline{2\pi r} \end{aligned}$$



Disk surface integral

when $\theta = \pi/2$, $\hat{\mathbf{k}} = -\hat{\boldsymbol{\theta}}$

$$d\mathbf{A} = -\hat{\boldsymbol{\theta}} r' dr' d\phi$$

$$(\nabla \times \mathbf{F}) \cdot d\mathbf{A} = dr' d\phi$$

$$\oint_D (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = \int_0^r dr' \int_0^{2\pi} d\phi = \underline{2\pi r}$$

Problem 2.27

Integral over sphere radius a

$$\underline{\mathbf{F}(\mathbf{x}) = \mathbf{x} / r^3 = \hat{\mathbf{r}} / r^2}$$

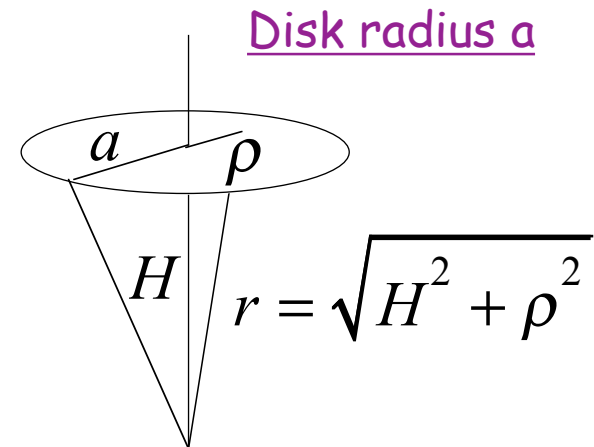
$$d\mathbf{A} = \hat{\mathbf{r}} r^2 \sin\theta d\theta d\phi$$

$$\oint_S \mathbf{F} \cdot d\mathbf{A} = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = 2\pi(-\cos\theta) \Big|_0^\pi = \underline{4\pi}$$

Prep for integral over Disk

$$d\mathbf{A} = \hat{\mathbf{k}} \rho d\rho d\phi = (\hat{\mathbf{r}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta) \rho d\rho d\phi$$

$$r = \sqrt{H^2 + \rho^2}; \quad \mathbf{F} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\hat{\mathbf{r}}}{H^2 + \rho^2}; \quad \cos\theta = \frac{H}{\sqrt{H^2 + \rho^2}}$$



Integral over Disk radius a

$$\begin{aligned} \oint_S \mathbf{F} \cdot d\mathbf{A} &= H \int_0^a \frac{\rho}{(H^2 + \rho^2)^{3/2}} d\rho \int_0^{2\pi} d\phi \\ &= \underline{2\pi \left(1 - \frac{H}{(H^2 + a^2)^{1/2}} \right)} \end{aligned}$$

Integral when H = a

$$\underline{\oint_S \mathbf{F} \cdot d\mathbf{A} = 2\pi \left(1 - \frac{1}{\sqrt{2}} \right)}$$