PHY481: Electromagnetism

Curl and divergence of E

& a bit more vector mathematics

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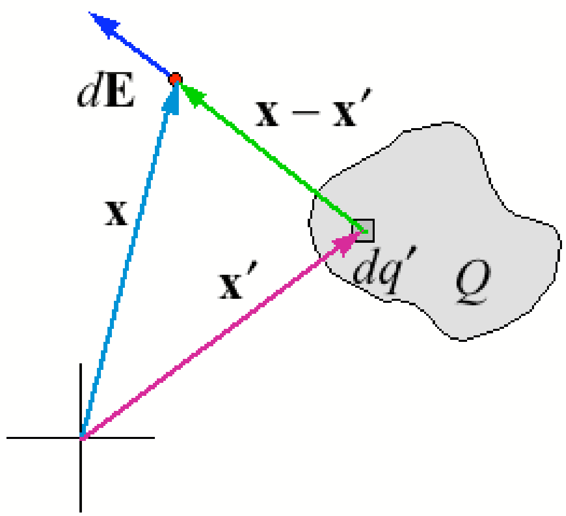
Electric field of charge distributions

- Charge distributions

Distribution:	Volume	Surface	Line
Charge density:	$\rho(\mathbf{x}')$	$\sigma(\mathbf{x}')$	$\lambda(\mathbf{x}')$
Total Charge:	$Q = \int \rho(\mathbf{x}')d^3x'$	$Q = \int \sigma(\mathbf{x}')d^2x'$	$Q = \int \lambda(\mathbf{x}')dx'$

- Integration over charge distribution

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')d^3x'$$

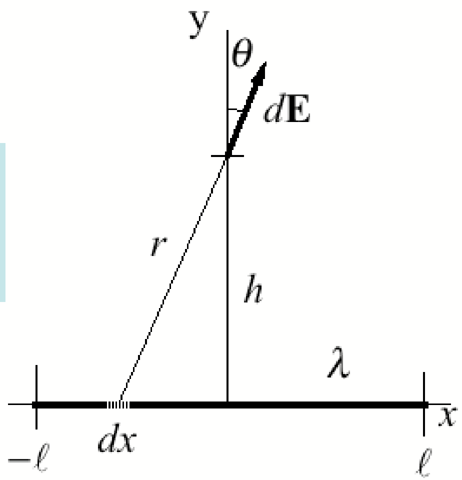


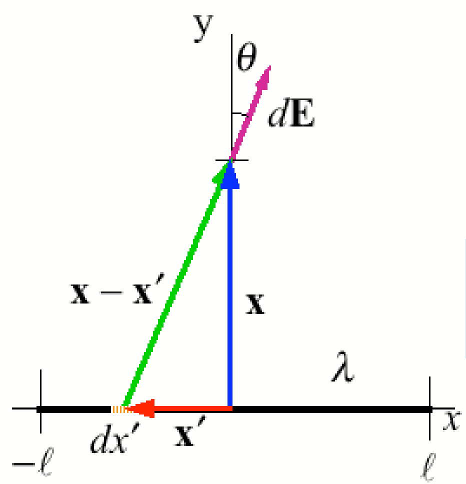
Old way   Line of charge   New way

$$r = \left(h^2 + x^2\right)^{1/2}$$

$$\cos\theta = \frac{h}{\left(h^2 + x^2\right)^{1/2}}$$

$$dq = \lambda dx$$





$$\mathbf{x} = h\hat{\mathbf{j}}$$

$$\mathbf{x}' = x'\hat{\mathbf{i}}$$

$$\mathbf{x} - \mathbf{x}' = h\hat{\mathbf{j}} - x'\hat{\mathbf{i}}$$

$$\left|\mathbf{x} - \mathbf{x}'\right| = \left(h^2 + x'^2\right)^{1/2}$$

$$\rho(\mathbf{x}')d^3x' = \lambda dx'$$

$$E_y = \int dE_y = \int dE\cos\theta = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r^2}\cos\theta$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{-\ell}^{\ell}\frac{\lambda dx}{\left(h^2 + x^2\right)}\cos\theta$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{-\ell}^{\ell}\frac{\lambda dx}{\left(h^2 + x^2\right)}\left(\frac{h}{\left(h^2 + x^2\right)^{1/2}}\right)$$

$$E_y = \frac{\lambda h}{4\pi\varepsilon_0}\int_{-\ell}^{\ell}\left(h^2 + x^2\right)^{-3/2} dx$$

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{x} - \mathbf{x}'}{\left|\mathbf{x} - \mathbf{x}'\right|^3}\rho(\mathbf{x}')d^3x'$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{-\ell}^{\ell}\frac{h\hat{\mathbf{j}} - x'\hat{\mathbf{i}}}{\left(h^2 + x'^2\right)^{3/2}}\lambda dx'$$

$$\mathbf{E} = \hat{\mathbf{j}}\frac{h\lambda}{4\pi\varepsilon_0}\int_{-\ell}^{\ell}\left(h^2 + x'^2\right)^{-3/2} dx'$$

Angular dependence automatically included !

Curl and divergence of E

- **Prove** differential form of Gauss's Law

Gauss's Law: $$\nabla \cdot \mathbf{E}(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\varepsilon_0}$$ True even for **E**(**x**,t)

with

$$\nabla \times \mathbf{E}(\mathbf{x}) = 0$$ True only for static charge

- Will need definition of the scalar potential function $V(\mathbf{x})$

$$V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{x}')d^3x'}{|\mathbf{x} - \mathbf{x}'|}$$

x = evaluation coordinates

x' = source positions

| **x** - **x**' | (not | |^2 or | |^3)

- Will need Gradient of V
  - Derivatives are with respect to x, not x': $\nabla = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}$
  - Bring Del into the integral over d^3x'

$$\nabla V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \nabla \left(\int \frac{\rho(\mathbf{x}')d^3x'}{|\mathbf{x} - \mathbf{x}'|} \right) = \frac{1}{4\pi\varepsilon_0} \int \rho(\mathbf{x}') \nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) d^3x'$$

Gradient of 1/|x-x'|

Since:

$$\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)=\frac{-1}{|\mathbf{x}-\mathbf{x}'|^2}\nabla|\mathbf{x}-\mathbf{x}'|$$

$$=-\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}$$

Using chain rule

$$\frac{\partial}{\partial x_i}f[g(x)]=f'[g(x)]\frac{\partial g(x)}{\partial x_i}$$

$$|\mathbf{x}-\mathbf{x}'|=\sqrt{(x_i-x_i')^2}$$

$$\nabla|\mathbf{x}-\mathbf{x}'|=\frac{2(x_i-x_i')\,\mathbf{e}_i}{2\sqrt{(x_i-x_i')^2}}=\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}$$

, then

$$\nabla V(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\int\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\rho(\mathbf{x}')d^3x'$$

$$=-\frac{1}{4\pi\varepsilon_0}\int\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')d^3x'$$

$$=-\mathbf{E}(\mathbf{x})$$

Presented as

$$\mathbf{E}(\mathbf{x})=-\nabla V(\mathbf{x})$$

Del squared of V

- Still trying to prove Gauss's Law !

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\varepsilon_0}$$

- We now have available $\mathbf{E}(\mathbf{x}) = -\nabla V(\mathbf{x})$

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = \nabla \cdot \left(-\nabla V\right) = -\nabla^2 V$$

(-) Laplacian of V

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = -\nabla^2 V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int -\nabla^2 \left[\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right] \rho(\mathbf{x}') d^3x'$$

- Two will be the same if we can show that

$$\int_{Vol.} -\nabla^2 \left[\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right] d^3x' = 4\pi \text{ if } Volume \text{ includes } \mathbf{x}' = \mathbf{x}$$

Involves Dirac's delta function

Laplacian of 1/|**x**-**x**'|

- To prove Gauss's Law we need to show

$$\int_{Vol.} -\nabla^2\left[\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right] d^3x' = 4\pi \text{ if } Volume \text{ includes } \mathbf{x}' = \mathbf{x}$$

- Point source at **x'** = 0, and use spherical coordinates **x** = **r**

$$\nabla^2\left[\frac{1}{|\mathbf{x}|}\right] = \nabla^2\left[\frac{1}{r}\right] = \frac{d^2}{dr^2}\left[\frac{1}{r}\right] + \frac{2}{r}\frac{d}{dr}\left[\frac{1}{r}\right]$$

$$= \frac{2}{r^3} - \frac{2}{r^3}$$ = 0 everywhere ?

What about at $r = 0$?

- Strategy to evaluate at the singular point, $r = 0$

Use Gauss's theorem & a little sphere radius R around $r = 0$

Strange Laplacian of 1/|x-x'|

- Value of $\nabla^2\left[\frac{1}{r}\right]=0$ But perhaps not at r = 0 !
- Use Gauss's theorem to evaluate at r = 0

$$\int_V \nabla\cdot\mathbf{F}\,d^3x=\int_S \mathbf{F}\cdot d\mathbf{A}$$

$$\text{with } \mathbf{F}=\nabla[1/r]$$

$$\int_V \nabla^2\left[\frac{1}{r}\right]d^3x=\int_S \nabla\left[\frac{1}{r}\right]\cdot\hat{\mathbf{r}}\,dA$$

$$\nabla\left[\frac{1}{r}\right]=\frac{\partial}{\partial r}\left[\frac{1}{r}\right]\hat{\mathbf{r}}=\frac{-1}{r^2}\hat{\mathbf{r}}$$

$$dA=R^2\sin\theta\,d\theta\,d\phi$$

- Evaluate <u>right side</u> with sphere, radius R around origin

$$\int_V \nabla^2\left[\frac{1}{r}\right]d^3x=\frac{-1}{R^2}\int_0^{\pi}R^2\sin\theta\,d\theta\int_0^{2\pi}d\phi=-4\pi$$

(pretty amazing)

Does not depend on sphere radius R !

$$\int_{Vol.} -\nabla^2\left[\frac{1}{r}\right]d^3x'=4\pi \text{ if } Volume \text{ includes } r=0$$

Dirac delta function

- Strange behavior of the Laplacian of 1/r

$$\int_{Vol.} -\nabla^2\left[\frac{1}{|\mathbf{x}|}\right] d^3x' = \begin{cases} 4\pi & \text{if } Volume \text{ includes } \mathbf{x}=0 \\ 0 & \text{elsewhere} \end{cases}$$

- Define Dirac delta function (very useful in advanced physics)

$$\delta^3(\mathbf{x}) = -\frac{1}{4\pi}\nabla^2\left[\frac{1}{|\mathbf{x}|}\right]$$

$$\int_{Vol.} \delta^3(\mathbf{x}) d^3x' = \begin{cases} 1 & \text{if } Volume \text{ includes } \mathbf{x}=0 \\ 0 & \text{elsewhere} \end{cases}$$

- Finally, we prove Gauss's Law

$$\nabla \cdot \mathbf{E} = -\nabla^2 V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int -\nabla^2\left[\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right]\rho(\mathbf{x}')d^3x'$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\int \delta^3(\mathbf{x}-\mathbf{x}')\rho(\mathbf{x}')d^3x' = \frac{\rho(\mathbf{x})}{\varepsilon_0}$$

delta function picks out value of ρ at point where $\mathbf{x} = \mathbf{x}'$

Green's functions

- Poisson's equation and a general solution

$$-\nabla^2 V(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\varepsilon_0} \qquad V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

- Green's function of a differential operator
  - The Laplacian is just one of many important differential operators
  - Consider a unit point source at **x**' $\qquad \rho_0(\mathbf{x}) = \delta^3(\mathbf{x}-\mathbf{x}')$
  - The Green's function G for the Laplacian (which we have just found) is the solution for the potential due a unit point source

$$-\nabla^2 G\left(\mathbf{x}-\mathbf{x}'\right) = \delta^3(\mathbf{x}-\mathbf{x}') \qquad G(\mathbf{x}-\mathbf{x}') = \frac{1}{4\pi\,|\mathbf{x}-\mathbf{x}'|}$$

- The potential V(**x**) for an arbitrary source distribution is then

$$V(\mathbf{x}) = \int G(\mathbf{x}-\mathbf{x}')\rho(\mathbf{x}')d^3x'$$

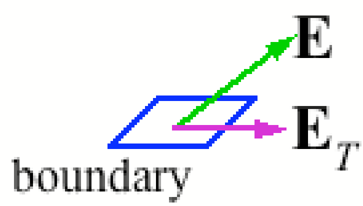
Stokes's theorem and E at a boundary

- Stokes's theorem

$$\oint_S (\nabla \times \mathbf{E}) \cdot \hat{\mathbf{n}}\, dS = \oint_C \mathbf{E} \cdot d\boldsymbol{\ell}$$

- Electric field has zero curl

$$\nabla \times \mathbf{E} = 0$$



$\mathbf{E}_T$ is tangential projection of **E**

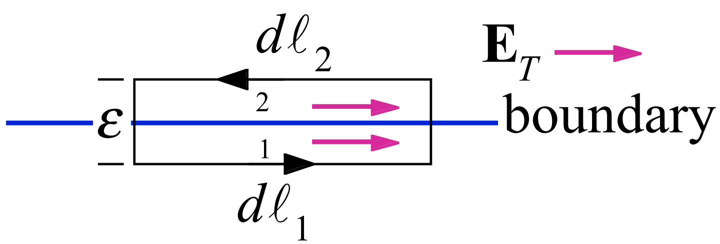
  - Loop near a boundary

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0$$

$$\mathbf{E}_{T1} \cdot d\boldsymbol{\ell}_1 + \mathbf{E}_{T2} \cdot d\boldsymbol{\ell}_2 = 0$$

$$(\mathbf{E}_{T1} - \mathbf{E}_{T2}) \cdot d\boldsymbol{\ell}_1 = 0$$

$$\mathbf{E}_{T1} = \mathbf{E}_{T2}$$



$$\varepsilon \to 0$$

$$d\boldsymbol{\ell}_2 \to -d\boldsymbol{\ell}_1 \quad \text{(vectors)}$$

Tangential components of E are continuous

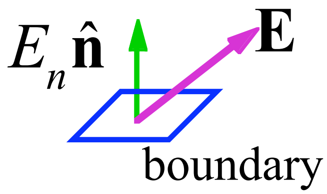
Gauss's theorem and E at a boundary

- Gauss's theorem $\oint_S \mathbf{E}\cdot d\mathbf{A} = \oint_V \nabla\cdot\mathbf{E}\, d^3x$ & law $\nabla\cdot\mathbf{E} = \rho/\varepsilon_0$

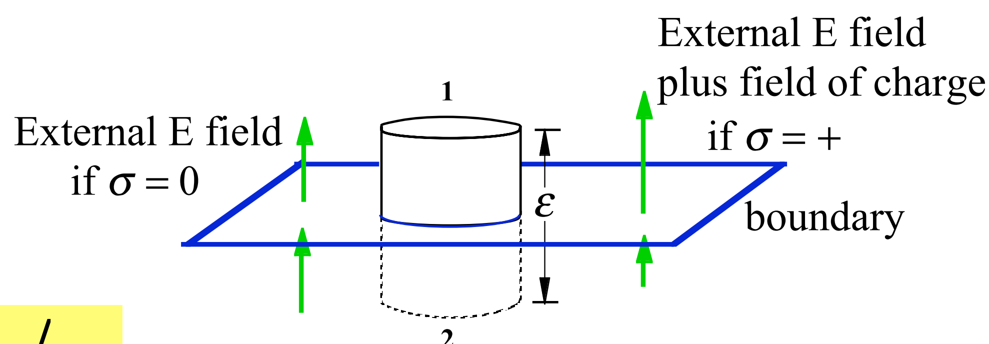
Integral form $\oint_S \mathbf{E}\cdot d\mathbf{A} = q_{encl}/\varepsilon_0$

$$\frac{1}{\varepsilon_0}\oint_V \rho(\mathbf{x})\,d^3x = \frac{q_{encl}}{\varepsilon_0}$$

E_n is normal component of **E**



- Crossing a boundary



$\varepsilon \to 0$

$\hat{\mathbf{n}}_2 = -\hat{\mathbf{n}}_1$

$\sigma = dq_{encl}/dA$

upper disk lower disk

$$\left(\mathbf{E}_1\cdot\hat{\mathbf{n}}_1 dA_1 + \mathbf{E}_2\cdot\hat{\mathbf{n}}_2 dA_2\right) = dq_{encl}/\varepsilon_0$$

$$\left(\mathbf{E}_1 - \mathbf{E}_2\right)\cdot\hat{\mathbf{n}}_1 dA =$$

$$\left(E_{n1} - E_{n2}\right) = \sigma/\varepsilon_0$$

Normal components of **E** differ by σ/ε_0.

But if $\sigma = 0$, E_n continuous