PHY481: Electromagnetism

Dirac delta function

E field near a boundary

Hints for solving HW problems

Dirac delta function

- Strange behavior of the Laplacian of 1/r

$$\int_{Vol.} -\nabla^2\left[\frac{1}{|\mathbf{x}|}\right] d^3x' = \begin{cases} 4\pi & \text{if } Volume \text{ includes } \mathbf{x}=0 \\ 0 & \text{elsewhere} \end{cases}$$

- Define Dirac delta function (very useful in advanced physics)

$$\delta^3(\mathbf{x}) = -\frac{1}{4\pi}\nabla^2\left[\frac{1}{|\mathbf{x}|}\right]$$

$$\int_{Vol.} \delta^3(\mathbf{x}) d^3x' = \begin{cases} 1 & \text{if } Volume \text{ includes } \mathbf{x}=0 \\ 0 & \text{elsewhere} \end{cases}$$

- Finally, we prove Gauss's Law

$$\nabla \cdot \mathbf{E} = -\nabla^2 V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int -\nabla^2\left[\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right]\rho(\mathbf{x}')d^3x'$$

delta function picks out value of ρ at point where $\mathbf{x} = \mathbf{x}'$

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\int \delta^3(\mathbf{x}-\mathbf{x}')\rho(\mathbf{x}')d^3x' = \frac{\rho(\mathbf{x})}{\varepsilon_0}$$

Dirac delta function & the Jacobian

- P3.17c. Prove: $$\int_{-\infty}^{\infty} \delta^3(\mathbf{A}\mathbf{x}+\mathbf{b}) f(\mathbf{x}) d^3x = f(-\mathbf{A}^{-1}\mathbf{b}) / \left|\det \mathbf{A}\right|$$

Change variables

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}; \quad x'_i = A_{ij}x_j + b_j$$

$$\mathbf{x} = -\mathbf{A}^{-1}\mathbf{b} \text{ when } \mathbf{x}' = 0$$

Jacobian

$$d^3x' = J d^3x \quad \text{where } J = \left|\det\left[\frac{\partial x'_i}{\partial x_k}\right]\right|$$

$$J = \left|\det\left(A_{ij}\frac{\partial x_j}{\partial x_k}\right)\right| = \left|\det A_{ij}\right|$$

Another Jacobian example:

$$d^3x = dxdydz = J\, dr d\theta d\phi$$

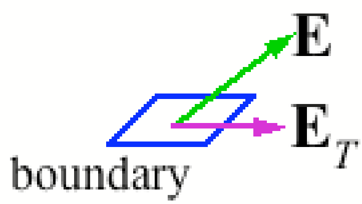
$$(x,y,z) = \left(r\sin\theta\cos\phi,\, r\sin\theta\sin\phi,\, r\cos\theta\right)$$

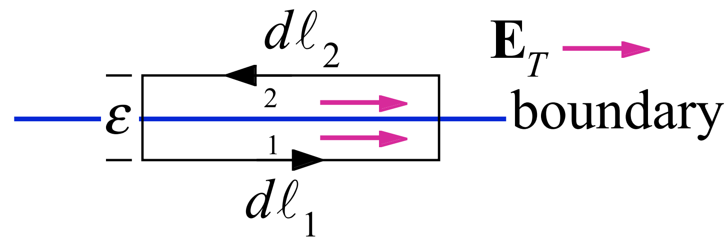
$$J = \left|\det\begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{pmatrix}\right| = r^2\sin\theta$$

Tangential and Normal Components of E

Summary

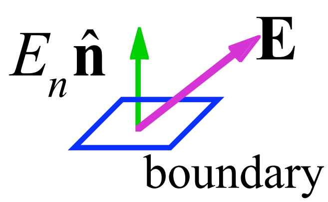
$\mathbf{E}_T$ is tangential projection of **E**

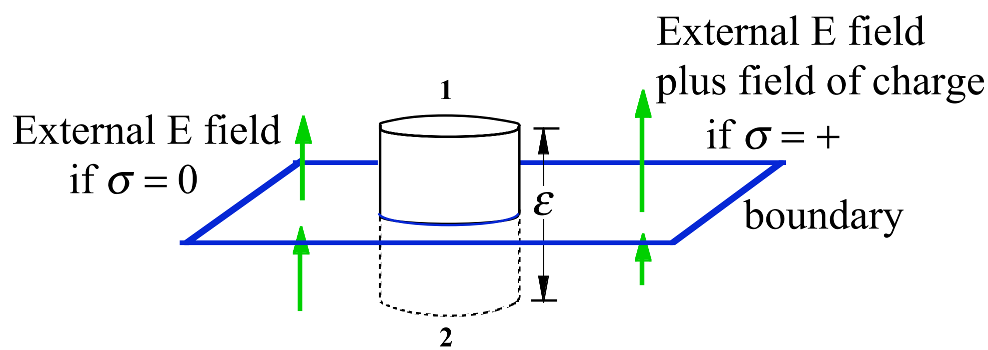




Tangential components of E are continuous

E_n is normal component of **E**





Normal components of E differ by σ/ε_0.

Problem 3.5

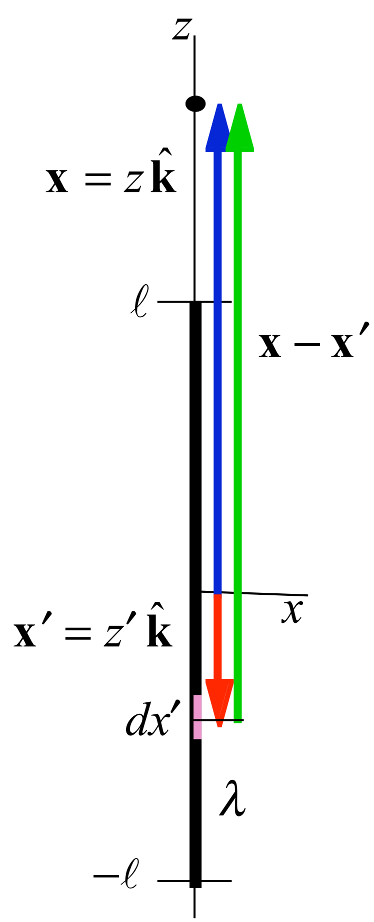
Vertical line of charge. Find electric field on z-axis above the charge.

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')d^3x'$$

$$\mathbf{x} = z\,\hat{\mathbf{k}}$$
$$\mathbf{x}' = z'\,\hat{\mathbf{k}}$$
$$\mathbf{x}-\mathbf{x}' = (z-z')\,\hat{\mathbf{k}}$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{-\ell}^{\ell} \frac{(z-z')\hat{\mathbf{k}}}{(z-z')^3}\lambda dz'$$

$$\rho(\mathbf{x}')d^3x' = \lambda dz'$$

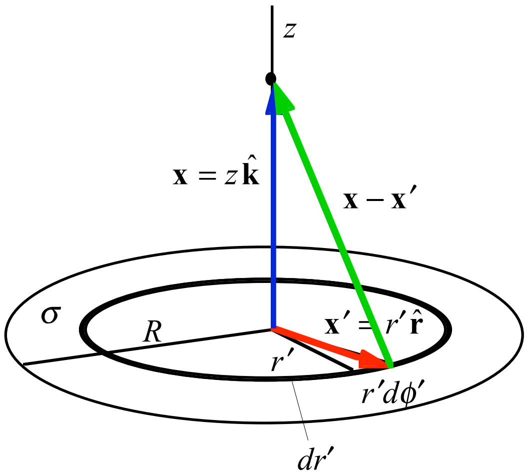


Problem 3.6

Disk of charge. Find electric field on z-axis.

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')d^3x'$$

$$\mathbf{x} = z\,\hat{\mathbf{k}}$$
$$\mathbf{x}' = r'\,\hat{\mathbf{r}}$$



$$= \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi \int_0^R \frac{z\,\hat{\mathbf{k}} - r'\,\hat{\mathbf{r}}}{(z^2+r'^2)^{3/2}} r'dr'$$

ϕ integration first

$$(\hat{\mathbf{r}} = \hat{\mathbf{i}}\cos\phi' + \hat{\mathbf{j}}\sin\phi')$$

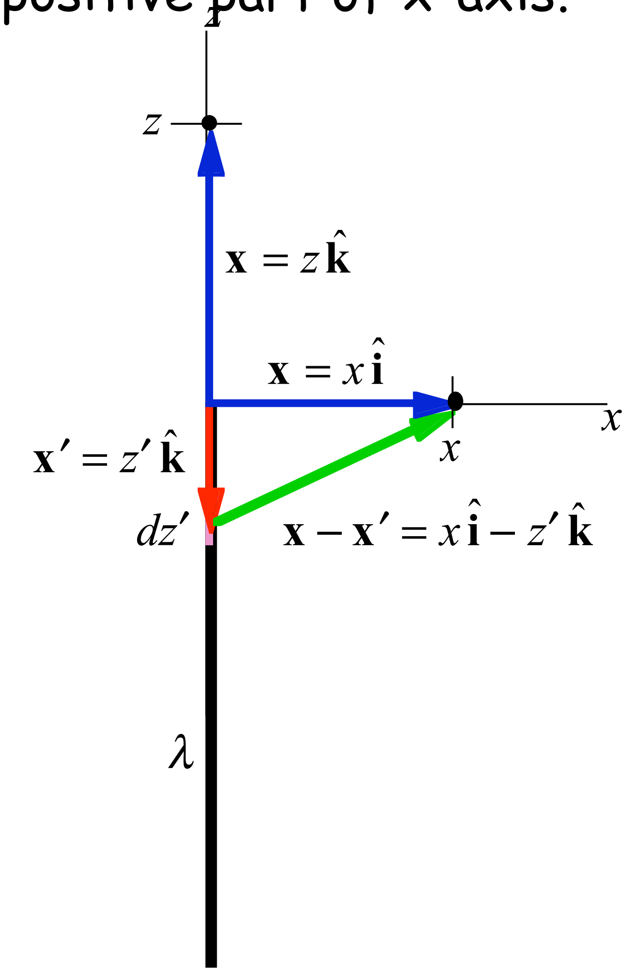
$$\mathbf{x}-\mathbf{x}' = z\,\hat{\mathbf{k}} - r'\,\hat{\mathbf{r}}$$
$$|\mathbf{x}-\mathbf{x}'| = \left(z^2 + r'^2\right)^{1/2}$$

$$\rho(\mathbf{x}')d^3x' = \sigma\, r'dr'd\phi'$$

Problem 3.8

Infinite line charge on negative part of z axis. Find electric field on a) positive part of z-axis, b) positive part of x-axis.

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')d^3x'$$



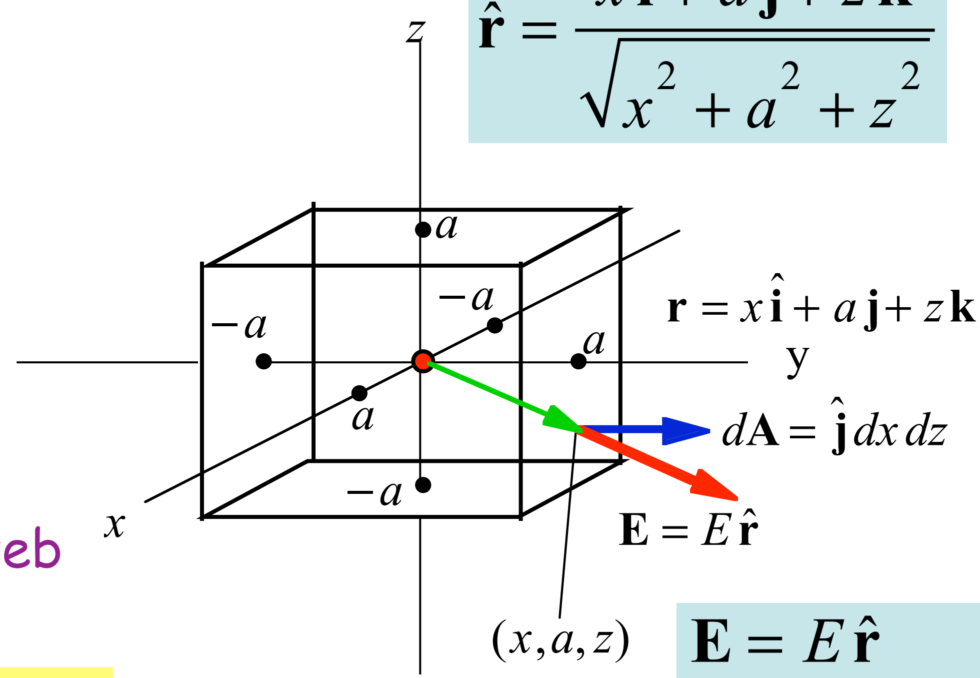
Problem 3.10

Charge q at the center of a box edge = $2a$. Check Gauss's law.

Gauss's law
$$\int_S \mathbf{E}\cdot d\mathbf{A} = \frac{q_{encl}}{\varepsilon_0}$$

Add up the flux through the 6 faces and check it adds up to the expected value

$$\hat{\mathbf{r}} = \frac{x\,\hat{\mathbf{i}} + a\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{x^2 + a^2 + z^2}}$$



You will need to look on the web for this integral

$$\int \frac{dx}{(a^2 + x^2)(2a^2 + x^2)} = \arctan[\quad]$$

$$\mathbf{E} = E\,\hat{\mathbf{r}}$$

$$E = \frac{q}{4\pi\varepsilon_0 r^2}$$

3.14

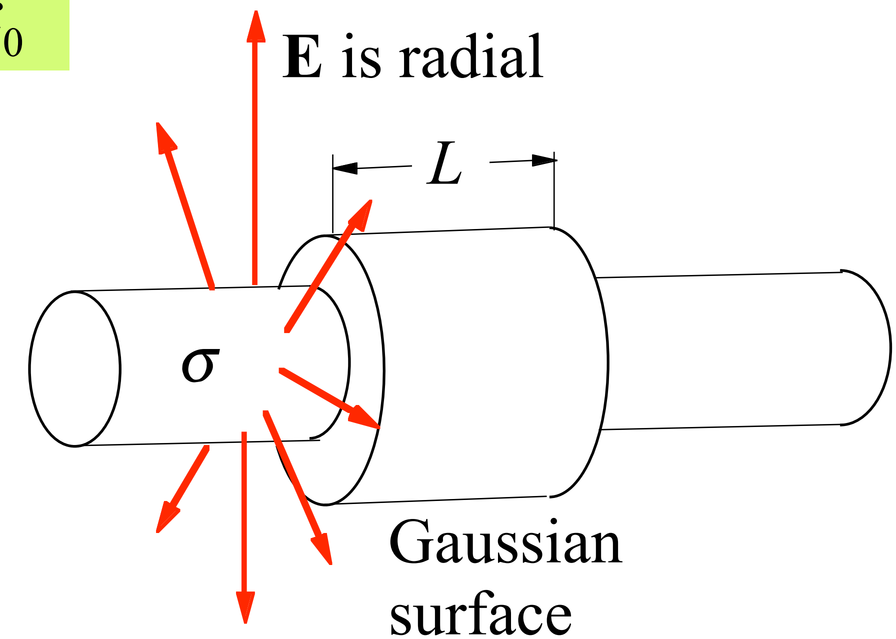
No help needed for this one!

3.15

Cylinder with uniform surface charge density. Use Gauss's law to determine the radial dependence of the field.

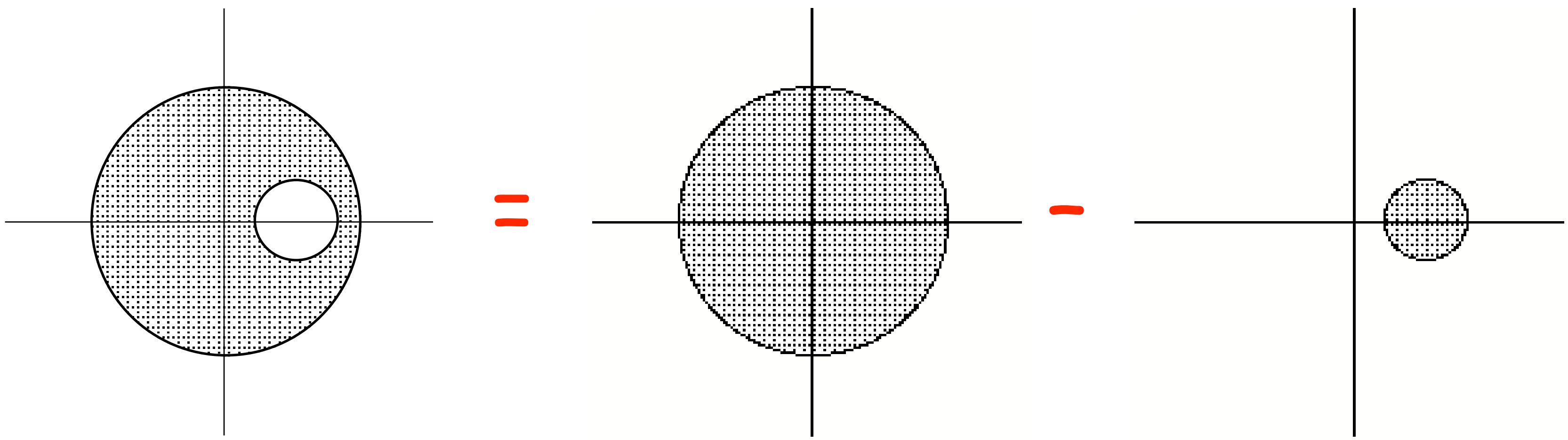
Gauss's law

$$\int_S \mathbf{E} \cdot d\mathbf{A} = \frac{q_{encl}}{\varepsilon_0}$$



3.42

Solid sphere with uniform charge density has a smaller sphere hollowed out. Find field in the cavity.



Determine the field inside a solid sphere using spherical coordinates.
Expect a horizontal field in the cavity, so do the subtraction using Cartesian coordinates.