PHY481: Electromagnetism

Dirac delta function
E field near a boundary
Hints for solving HW problems
Dirac delta function

- Strange behavior of the Laplacian of $1/r$
  \[
  \int_{\text{Vol.}} -\nabla^2 \left[ \frac{1}{|x|} \right] d^3 x' = \begin{cases} 
    4\pi & \text{if } \text{Volume includes } x = 0 \\
    0 & \text{elsewhere}
  \end{cases}
  \]

- Define Dirac delta function (very useful in advanced physics)
  \[
  \delta^3(x) = -\frac{1}{4\pi} \nabla^2 \left[ \frac{1}{|x|} \right]
  \]
  \[
  \int_{\text{Vol.}} \delta^3(x) d^3 x' = \begin{cases} 
    1 & \text{if } \text{Volume includes } x = 0 \\
    0 & \text{elsewhere}
  \end{cases}
  \]

- Finally, we prove Gauss's Law
  \[
  \nabla \cdot E = -\nabla^2 V(x) = \frac{1}{4\pi\varepsilon_0} \int_{\text{Vol.}} -\nabla^2 \left[ \frac{1}{|x - x'|} \right] \rho(x')d^3 x'
  \]
  \[
  \nabla \cdot E = \frac{1}{\varepsilon_0} \int \delta^3(x - x') \rho(x') d^3 x' = \frac{\rho(x)}{\varepsilon_0}
  \]

delta function picks out value of $\rho$ at point where $x = x'$
Dirac delta function & the Jacobian

- P3.17c. Prove:
  \[ \int_{-\infty}^{\infty} \delta^3(Ax + b)f(x)d^3x = f(-A^{-1}b)/|\det A| \]

Change variables:
- \( x' = Ax + b; \quad x'_i = A_{ij}x_j + b_j \)
- \( x = -A^{-1}b \) when \( x' = 0 \)

Jacobian:
- \( d^3x' = Jd^3x \) where \( J = \left| \det \left[ \frac{\partial x'_i}{\partial x'_k} \right] \right| \)
- \( J = \left| \det \left( A_{ij} \frac{\partial x_j}{\partial x_k} \right) \right| = |\det A_{ij}| \)

Another Jacobian example:
- \( d^3x = dx dy dz = J \, dr \, d\theta \, d\phi \)
- \((x, y, z) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta) \)
- \( J = \left| \det \begin{pmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ r \cos \theta \cos \phi & r \cos \theta \sin \phi & -r \sin \theta \\ -r \sin \theta \sin \phi & r \sin \theta \cos \phi & 0 \end{pmatrix} \right| = r^2 \sin \theta \)
Tangential and Normal Components of $E$

**Summary**

$E_T$ is tangential projection of $E$

Tangential components of $E$ are continuous

$E_n$ is normal component of $E$

Normal components of $E$ differ by $\sigma/\varepsilon_0$. 

$E_T$ is tangential projection of $E$

$E_n \hat{n}$ is normal component of $E$

$E_T$ is tangential projection of $E$

$E_n \hat{n}$ is normal component of $E$

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$E_T$ is tangential projection of $E$

$E_n \hat{n}$ is normal component of $E$
Problem 3.5

Vertical line of charge. Find electric field on z-axis above the charge.

\[
E = \frac{1}{4\pi \varepsilon_0} \int \frac{x - x'}{|x - x'|^3} \rho(x') d^3x' \\
= \frac{1}{4\pi \varepsilon_0} \int_{-\ell}^{\ell} \frac{(z - z')\hat{k}}{(z - z')^3} \lambda dz'
\]

\[
x = z\hat{k} \\
x' = z'\hat{k} \\
x - x' = (z - z')\hat{k}
\]

\[
\rho(x') d^3x' = \lambda dz'
\]
Problem 3.6

Disk of charge. Find electric field on z-axis.

\[
E = \frac{1}{4\pi\varepsilon_0} \int \frac{x - x'}{|x - x'|^3} \rho(x')d^3x' \\
= \frac{\sigma}{4\pi\varepsilon_0} \int_0^{2\pi} d\phi \int_0^R \frac{z\hat{k} - r'\hat{r}}{(z^2 + r'^2)^{3/2}} r'd\phi'
\]

\( \hat{r} = \hat{i}\cos\phi' + j\sin\phi' \)

**ϕ integration first**

\( \hat{r} = \hat{i}\cos\phi' + j\sin\phi' \)

\[
x - x' = z\hat{k} - r'\hat{r}
\]

\[
|x - x'| = \left(z^2 + r'^2\right)^{1/2}
\]

\[
\rho(x')d^3x' = \sigma r'dr'd\phi'
\]
Problem 3.8

Infinite line charge on negative part of z axis. Find electric field on a) positive part of z-axis, b) positive part of x-axis.

\[ E = \frac{1}{4\pi \varepsilon_0} \int \frac{\mathbf{x} - \mathbf{x}' \cdot \rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d^3 x' \]

\[ \mathbf{x} = z \hat{k} \]

\[ \mathbf{x}' = z' \hat{k} \]

\[ x - x' = x \hat{i} - z' \hat{k} \]
Problem 3.10

Charge $q$ at the center of a box edge $= 2a$. Check Gauss’s law.

Gauss's law

\[ \int_S \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{enc}}}{\varepsilon_0} \]

Add up the flux through the 6 faces and check it adds up to the expected value.

You will need to look on the web for this integral.

\[ \int \frac{dx}{(a^2 + x^2)(2a^2 + x^2)} = \arctan[ \ ] \]
No help needed for this one!
Cylinder with uniform surface charge density. Use Gauss’s law to determine the radial dependence of the field.

\[ \int_S \mathbf{E} \cdot d\mathbf{A} = \frac{q_{encl}}{\varepsilon_0} \]

Gauss’s law

\[ \mathbf{E} \text{ is radial} \]

Gaussian surface
Solid sphere with uniform charge density has a smaller sphere hollowed out. Find field in the cavity.

Determine the field inside a solid sphere using spherical coordinates. Expect a horizontal field in the cavity, so do the subtraction using Cartesian coordinates.