PHY481: Electromagnetism

HW3a problems
Potential V(x)

Problem 3.3

In an electric field E_0 in the z direction, a proton travels with an initial velocity v_0 in the x direction. What is its trajectory?

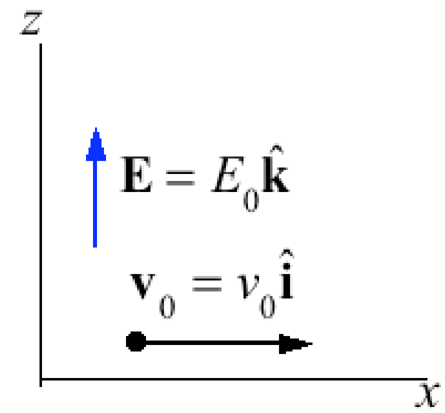
Acceleration: $$\mathbf{a} = \frac{\mathbf{F}}{m} = \frac{q\mathbf{E}}{m} = \frac{qE_0}{m}\hat{\mathbf{k}}$$

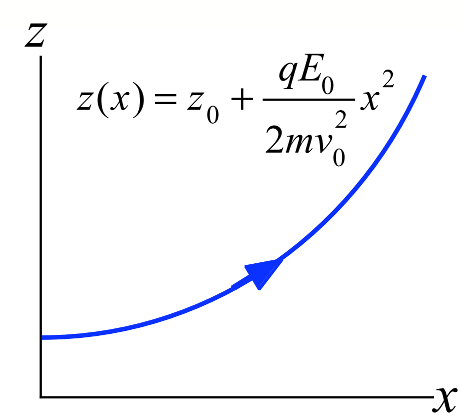
Displacement: $$\mathbf{s}(t) = \mathbf{v}_0 t + \tfrac{1}{2}\mathbf{a}t^2$$

Position: $$\mathbf{x}(t) = (x_0 + v_0 t)\hat{\mathbf{i}} + \left(z_0 + \frac{qE_0 t^2}{2m}\right)\hat{\mathbf{k}}$$

Trajectory: $$z(x) = z_0 + \frac{qE_0}{2mv_0^2}(x - x_0)^2$$

(Parabola)





Problem 3.5

Vertical line of charge. Find electric field on z-axis above the charge.

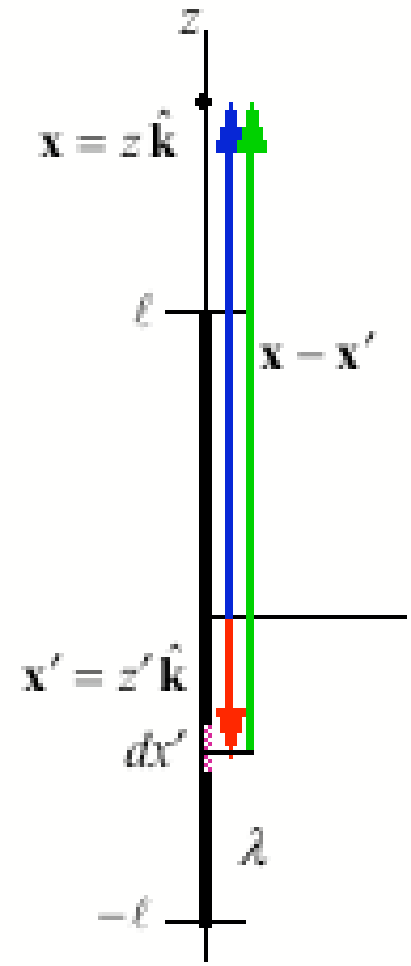
$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')d^3x'$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{-\ell}^{\ell} \frac{(z-z')\hat{\mathbf{k}}}{(z-z')^3}\lambda dz'$$

$$\mathbf{E} = \frac{\lambda\ell}{2\pi\varepsilon_0(z^2-\ell^2)}\hat{\mathbf{k}}$$

$$\mathbf{x} = z\,\hat{\mathbf{k}}$$
$$\mathbf{x}' = z'\,\hat{\mathbf{k}}$$
$$\mathbf{x}-\mathbf{x}' = (z-z')\,\hat{\mathbf{k}}$$

$$\rho(\mathbf{x}')d^3x' = \lambda dz'$$



For large z:

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0(z^2-\ell^2)}\hat{\mathbf{k}} \to \frac{Q}{4\pi\varepsilon_0 z^2}\hat{\mathbf{k}}\,,\; z >> \ell$$

E-field of a point charge $Q = 2\ell\lambda$

Problem 3.6

Disk of charge. Find electric field on z-axis.

$$\mathbf{E}=\frac{1}{4\pi\varepsilon_0}\int\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')d^3x'$$

ϕ' integration first

$$=\frac{\sigma}{4\pi\varepsilon_0}\int_0^R r'dr'\int_0^{2\pi}\frac{z\hat{\mathbf{k}}-r'\hat{\mathbf{r}}}{(z^2+r'^2)^{3/2}}d\phi' \qquad (\hat{\mathbf{r}}=\hat{\mathbf{i}}\cos\phi'+\hat{\mathbf{j}}\sin\phi')$$

$$E_z=\frac{\sigma z}{2\varepsilon_0}\int_0^R\frac{r'dr'}{(z^2+r'^2)^{3/2}}=\frac{\sigma z}{2\varepsilon_0}\left[-\frac{1}{\left(z^2+r'^2\right)^{1/2}}\right]_0^R$$

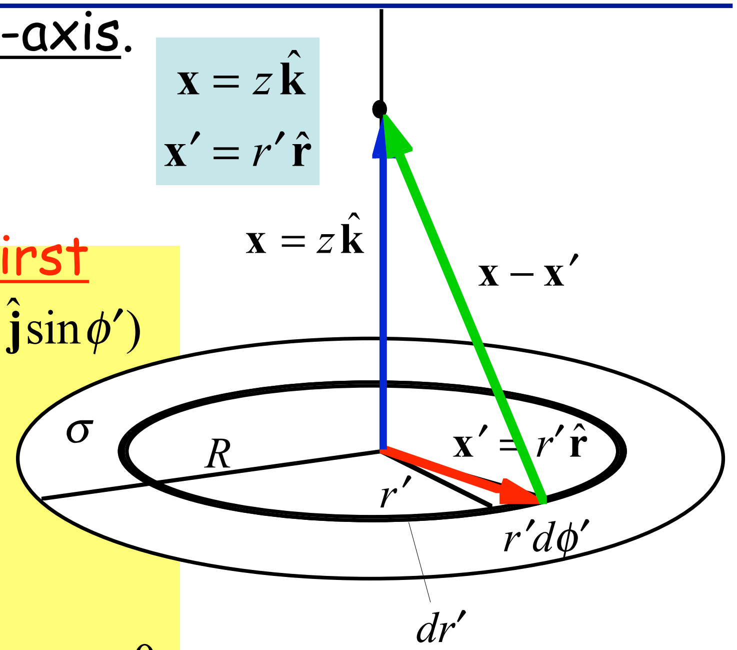
$$E_z=\underline{\frac{\sigma}{2\varepsilon_0}\left[\frac{z}{|z|}-\frac{z}{\left(z^2+R^2\right)^{1/2}}\right]}\approx\underline{\frac{\sigma}{2\varepsilon_0}\text{(infinite sheet); } R>>z>0}$$

$$=\frac{2\pi\sigma}{4\pi\varepsilon_0}\left[1-\frac{1}{\left(1+\frac{R^2}{z^2}\right)^{1/2}}\right]\approx\frac{2\pi\sigma}{4\pi\varepsilon_0}\left(\frac{R^2}{2z^2}\right),\quad z>0$$

$$E_z=\underline{\frac{q}{4\pi\varepsilon_0 z^2}\text{(point charge); } z>>R}$$

$\mathbf{x}=z\hat{\mathbf{k}}$

$\mathbf{x}'=r'\hat{\mathbf{r}}$



$\mathbf{x}-\mathbf{x}'=z\hat{\mathbf{k}}-r'\hat{\mathbf{r}}$

$|\mathbf{x}-\mathbf{x}'|=\left(z^2+r'^2\right)^{1/2}$

$\rho(\mathbf{x}')d^3x'=\sigma r'dr'd\phi'$

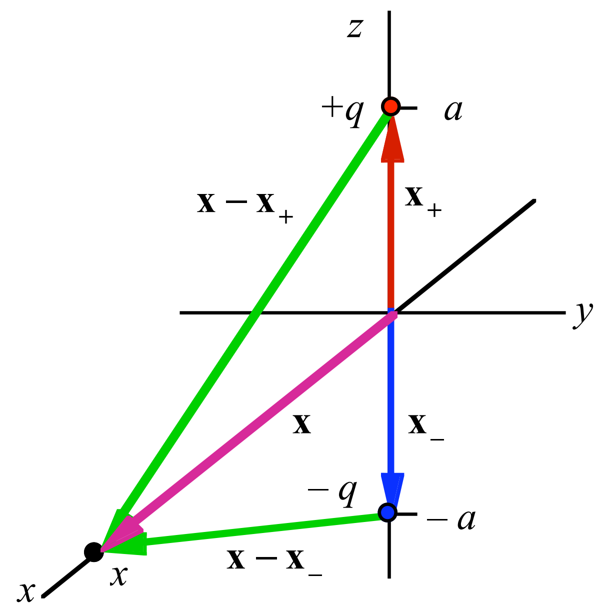
Problem 3.7

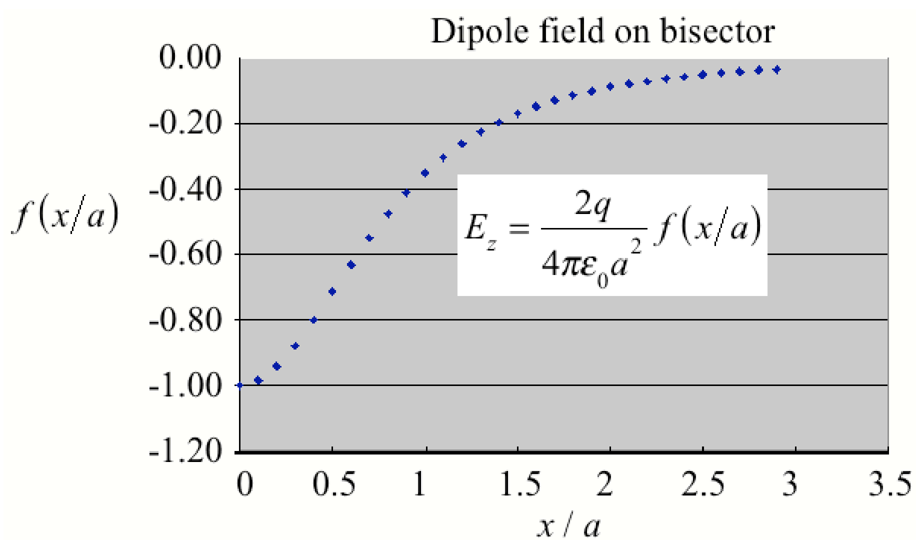
$$\mathbf{E}(x) = \frac{q}{4\pi\varepsilon_0}\left[\frac{x\,\hat{\mathbf{i}} - a\,\hat{\mathbf{k}}}{\left(x^2 + a^2\right)^{3/2}} - \frac{x\,\hat{\mathbf{i}} + a\,\hat{\mathbf{k}}}{\left(x^2 + a^2\right)^{3/2}}\right]$$

$$E_x(x,0,0) = \underline{0}$$

$$E_z(x,0,0) = -\frac{q}{2\pi\varepsilon_0 a^2}\frac{1}{\left(1 + x^2/a^2\right)^{3/2}}$$

$$\approx \underline{-\frac{2q}{4\pi\varepsilon_0 a^2}\ |x| << a;} \qquad \underline{-\frac{2qa}{4\pi\varepsilon_0 x^3}\ |x| >> a}$$



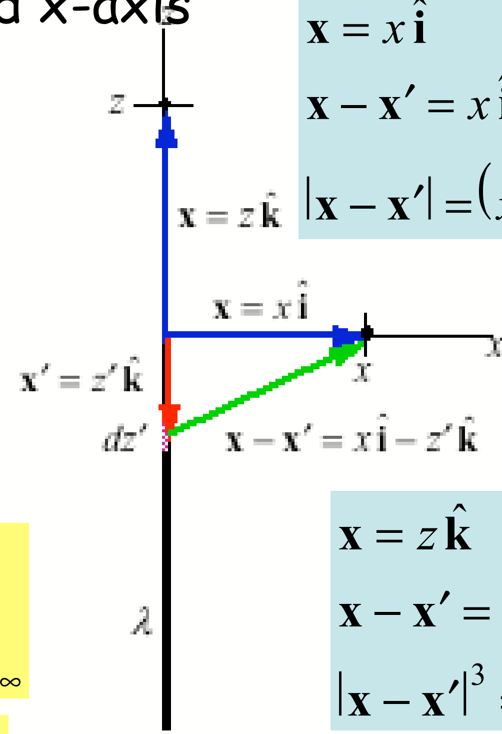


Problem 3.8

Infinite line charge on negative part of z axis. Find electric field on positive part of the z-axis, and x-axis

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{x}-\mathbf{x}'}{\left|\mathbf{x}-\mathbf{x}'\right|^3}\rho(\mathbf{x}')d^3x'$$

$$\mathbf{x} = x\,\hat{\mathbf{i}}$$
$$\mathbf{x}-\mathbf{x}' = x\,\hat{\mathbf{i}} - z'\,\hat{\mathbf{k}}$$
$$\left|\mathbf{x}-\mathbf{x}'\right| = \left(x^2+z'^2\right)^{1/2}$$



<u>z-axis</u>

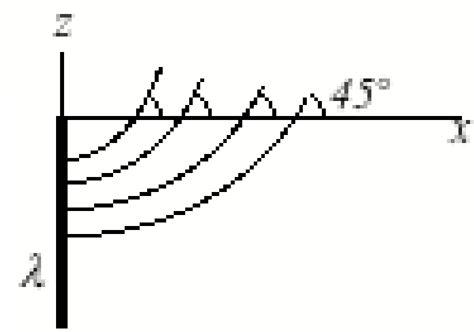
$$E_z(0,0,z) = \frac{\lambda}{4\pi\varepsilon_0}\int_{-\infty}^{0}\frac{dz'}{(z-z')^2} = \frac{\lambda}{4\pi\varepsilon_0}\left.\frac{1}{\xi}\right|_{-\infty}^{z} = \underline{\frac{\lambda}{4\pi\varepsilon_0 z}}$$

$$\mathbf{x} = z\,\hat{\mathbf{k}}$$
$$\mathbf{x}-\mathbf{x}' = \left(z-z'\right)\hat{\mathbf{k}}$$
$$\left|\mathbf{x}-\mathbf{x}'\right|^3 = \left(z-z'\right)^3$$

<u>x-axis</u>

$$E_x(x,0,0) = \frac{\lambda x}{4\pi\varepsilon_0}\int_{-\infty}^{0}\frac{dz'}{\left(x^2+z'^2\right)^{3/2}} = \frac{\lambda}{4\pi\varepsilon_0 x}\left[\frac{z'}{\left(x^2+z'^2\right)^{1/2}}\right]_{-\infty}^{0}$$

$$E_z(x,0,0) = \frac{\lambda}{4\pi\varepsilon_0}\int_{-\infty}^{0}\frac{-z'dz'}{\left(x^2+z'^2\right)^{3/2}} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{\left(x^2+z'^2\right)^{1/2}}\right]_{-\infty}^{0}$$



$$E_z(x,0,0) = E_x(x,0,0) = \underline{\frac{\lambda}{4\pi\varepsilon_0 x}}$$

$$\tan\theta = \frac{E_z}{E_x} = 1\ ;\ \theta = 45^\circ$$

Problem 3.10

Charge q at the center of a box edge = $2a$. Check Gauss's law.

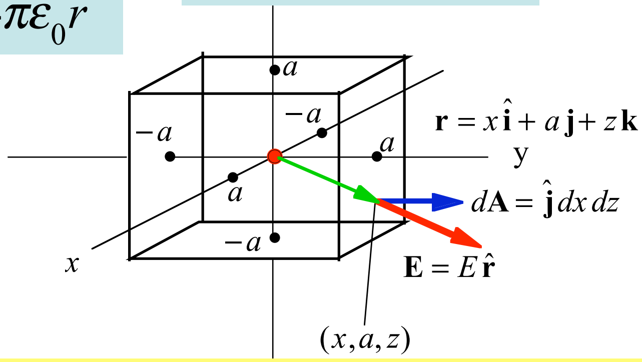
Gauss's law: $\int_S \mathbf{E}\cdot d\mathbf{A} = \dfrac{q_{encl}}{\varepsilon_0}$

$\mathbf{E} = E\,\hat{\mathbf{r}}$

$E = \dfrac{q}{4\pi\varepsilon_0 r^2}$

$\hat{\mathbf{r}} = \dfrac{x\,\hat{\mathbf{i}} + a\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}}{\sqrt{x^2 + a^2 + z^2}}$

Flux through right face (y = a)

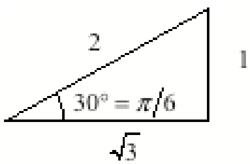


$$\oint_R \mathbf{E}\cdot d\mathbf{A} = \oint_R E\hat{\mathbf{r}}\cdot\hat{\mathbf{j}} = \frac{qa}{4\pi\varepsilon_0}\int_{-a}^{+a} dx \int_{-a}^{+a} dz \frac{1}{(x^2+a^2+z^2)^{3/2}}$$

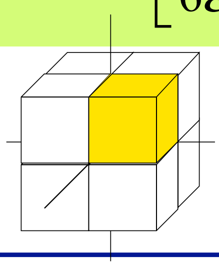
$$= \frac{qa}{4\pi\varepsilon_0}\int_{-a}^{+a} dx \left[\frac{za}{(a^2+x^2)(a^2+x^2+z^2)^{1/2}}\right]_{-a}^{+a}$$

$$= \frac{qa^2}{2\pi\varepsilon_0}\int_{-a}^{+a} dx \left[\frac{1}{(a^2+x^2)(2a^2+x^2)^{1/2}}\right] = \frac{q}{2\pi\varepsilon_0}\tan^{-1}\left[\frac{x}{(2a^2+x^2)^{1/2}}\right]_{-a}^{+a} = \frac{q}{\pi\varepsilon_0}\tan^{-1}\left[1/\sqrt{3}\right]$$

$$= \frac{q}{\pi\varepsilon_0}(\pi/6) = \frac{q}{6\varepsilon_0} \quad \text{and } 6\left[\frac{q}{6\varepsilon_0}\right] = \underline{\frac{q}{\varepsilon_0} \text{ through 6 faces}}$$



Cube with charge q in the corner

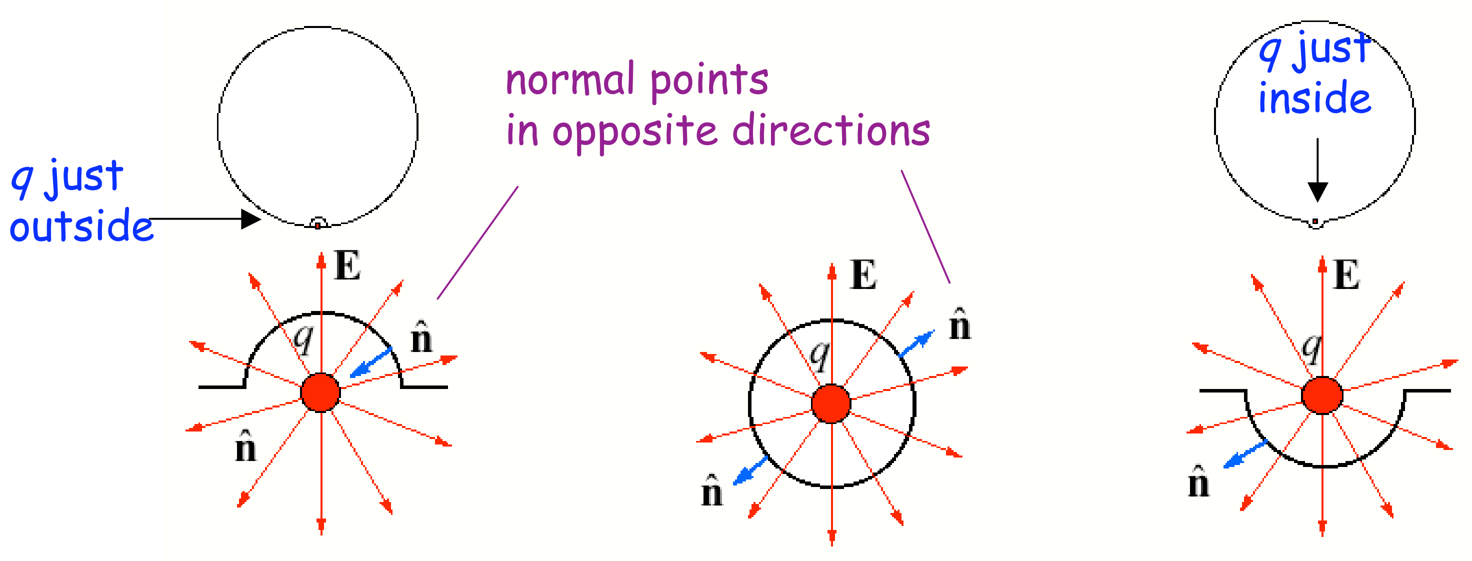


Flux through box is $\oint_{Corner} \mathbf{E}\cdot d\mathbf{A} = \dfrac{1}{8}\dfrac{q}{\varepsilon_0}$

Gauss's law problem (similar to Prob. 3.12)

Charge q, just inside or just outside a spherical surface.

Show that the flux through the surface is discontinuous by q/ε_0



$$\oint \mathbf{E}\cdot d\mathbf{A} = X \qquad \oint \mathbf{E}\cdot d\mathbf{A} = q/\varepsilon_0 \qquad \oint \mathbf{E}\cdot d\mathbf{A} = Y$$

$$\boxed{X + q/\varepsilon_0 = Y}$$

3.15

Cylinder with uniform surface charge density. Use Gauss's law to determine the radial dependence of the field.

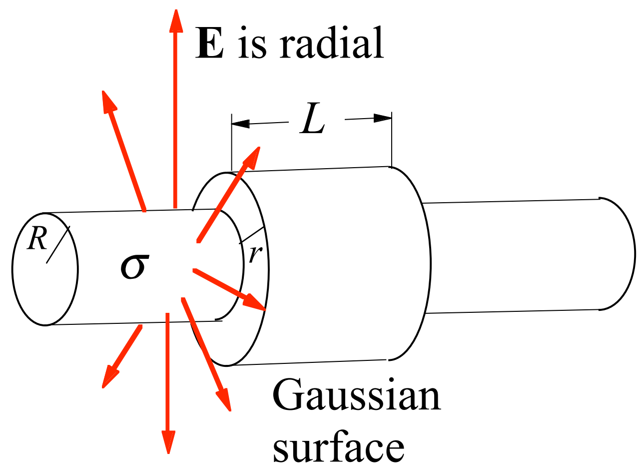
Gauss's law
$$\oint_S \mathbf{E}\cdot d\mathbf{A} = \frac{q_{encl}}{\varepsilon_0}$$

$$E_r A_r = \frac{A_R \sigma}{\varepsilon_0}$$

$$E_r 2\pi r L = \frac{2\pi R L \sigma}{\varepsilon_0}$$

$$\underline{E_r = \frac{\sigma}{\varepsilon_0}\frac{R}{r}, r > R;} \qquad E_r = \underline{0, r < R}$$

$$\mathbf{E} = E\,\hat{\mathbf{r}} \quad d\mathbf{A} = 2\pi r L\,\hat{\mathbf{r}}$$

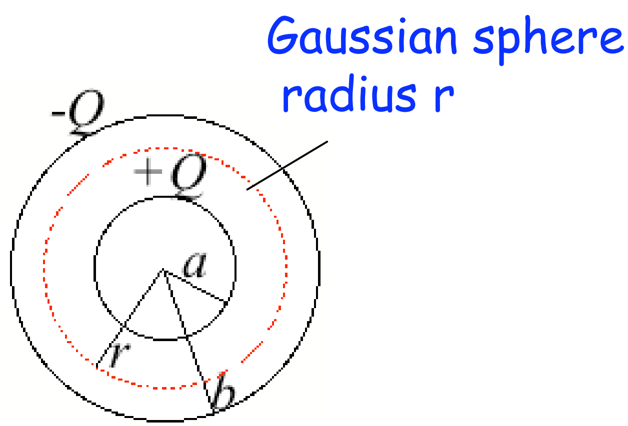


Disks of Gaussian surface do not contribute to integral

Problem 3.16

Concentric spherical shells charged *Q* and –*Q* with radii *a* and *b*. What is the field between the shells? What if –*Q* --> 0?

Gauss's law $\oint_S \mathbf{E}\cdot d\mathbf{A} = \frac{q_{encl}}{\varepsilon_0}$



$$E_r 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$\underline{E_r = \frac{Q}{4\pi\varepsilon_0 r^2} \text{ and } \mathbf{E}(\mathbf{x}) = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \quad a < r < b}$$

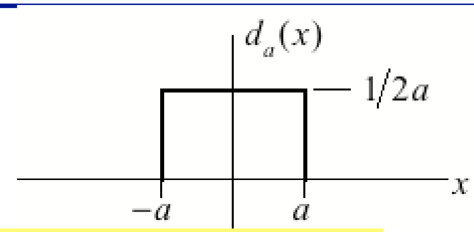
Point charge field

–*Q* --> 0 means -*Q* placed at infinity! Electric field between radii does not change, however, field now extends to infinite radius.

$$\underline{E_r = \frac{Q}{4\pi\varepsilon_0 r^2} \text{ and } \mathbf{E}(\mathbf{x}) = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \quad a < r}$$

Problem 3.19

Dirac δ function: $\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$



a) d_a(x) given by function in the figure. Let $f(x) = 1$

$$\int_{-\infty}^{\infty} d_a(x) f(x) dx = \int_{-\infty}^{\infty} d_a(x) dx = \frac{1}{2a} 2a = 1, \text{ and } \lim_{a \to 0} \left[d_a(x) \right] = \infty \;\; \Rightarrow d_a(x) = \delta(x)$$

b) $d_a(\mathbf{x}) = 1/V$ for $r \le a$, and 0 for $r > a$.

$$\int d_a(\mathbf{x}) d^3x = \frac{1}{V_a} \int_0^a d^3x = \frac{V_a}{V_a} = 1, \text{ and } \lim_{a \to 0} \left[d_a(\mathbf{x}) \right] = \infty \;\; \Rightarrow d_a(\mathbf{x}) = \delta(\mathbf{x})$$

c) Potential function V for uniformly charged sphere

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \oint_V \rho(\mathbf{x}') d^3x' : \quad E_{r>a} = \frac{Q}{4\pi\varepsilon_0 r^2}; \quad E_{r<a} 4\pi r^2 = \frac{Q}{\varepsilon_0} \frac{r^3}{a^3}, \quad E_{r<a} = \frac{Q}{4\pi\varepsilon_0} \frac{r}{a^3}$$

$$V_{r>a} = -\frac{Q}{4\pi\varepsilon_0} \int_\infty^r \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0 r}; \quad V_{r<a} = -\frac{Q}{4\pi\varepsilon_0} \int_\infty^a \frac{dr}{r^2} - \frac{Q}{4\pi\varepsilon_0 a^3} \int_a^r r dr = \frac{Q}{4\pi\varepsilon_0} \left[-\frac{r^2}{2a^3} + \frac{3}{2a} \right]$$

boundary
term

Problem 3.19 (cont)

d) V satisfies Laplaces equation for $r > a$, and Poisson's equation for $r < a$

$$-\nabla^2 V_{r>a} = \frac{Q}{4\pi\varepsilon_0}\left[\nabla^2 \frac{1}{r} = \frac{\partial}{r^2\partial r}\left(-r^2 \frac{1}{r^2}\right)\right] = 0$$

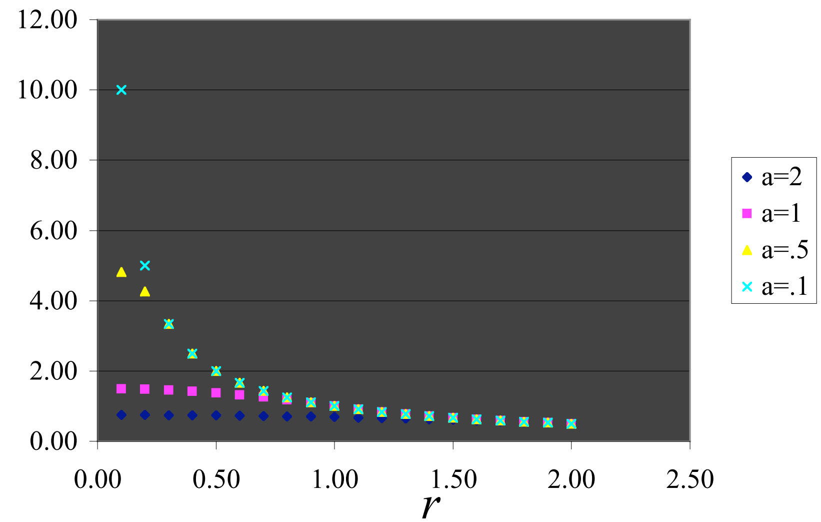
$$-\nabla^2 V_{r<a} = \frac{-Q}{4\pi\varepsilon_0 a^3}\left[\nabla^2\left(-\frac{r^2}{2}\right) = \frac{\partial}{r^2\partial r}\left(-r^2 r\right)\right] = \frac{Q}{\frac{4}{3}\pi a^3 \varepsilon_0} = \frac{\rho}{\varepsilon_0}$$

e) Show $V(\mathbf{x})$ approaches $Q / 4\pi\varepsilon_0 r$ as $a \to 0$

$$V(r,a) = \frac{Q}{4\pi\varepsilon_0} f(r,a)$$

$$f(r,a) = \frac{1}{r}\left[-\frac{r^3}{2a^3} + \frac{3r}{2a}\right]$$

$V(r,a)$ for $a = 2.0, 1.0, 0.5, 0.1$



Problem 3.24

Is this field irrotational? What is potential function, and Divergence?

$$\mathbf{E}(\mathbf{x}) = \hat{\mathbf{i}}\left(2x^2 - 2xy - 2y^2\right) + \hat{\mathbf{j}}\left(-x^2 - 4xy + y^2\right)$$

$$(\nabla \times \mathbf{E})_i = \varepsilon_{ijk}\frac{\partial E_k}{\partial x_j} \quad \text{no } j = 3 \text{ or } k = 3 \text{ terms}$$

$$(\nabla \times \mathbf{E})_3 = \frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} = (-2x - 4y) - (-2x - 4y) = 0$$

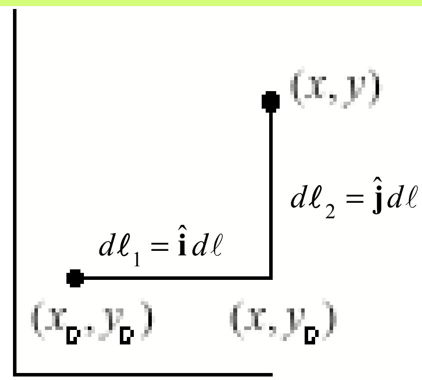
irrotational

$$V(\mathbf{x}) - V(\mathbf{x}_0) = -\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{E}\cdot d\ell$$

$$V(\mathbf{x}) = -\int_{x_0}^{x}\left(2x^2 - 2xy_0 - 2y_0^{\,2}\right)dx - \int_{y_0}^{y}\left(-x^2 - 4xy + y^2\right)dy$$

$$= -\left(\frac{2x^3}{3} - x^2y_0 - 2xy_0^{\,2}\right) + \left(\frac{2x_0^{\,3}}{3} - x_0^{\,2}y_0 - 2x_0y_0^{\,2}\right)$$

$$-\left(-x^2y - 2xy^2 + \frac{y^3}{3}\right) + \left(-x^2y_0 - 2xy_0^{\,2} + \frac{y_0^{\,3}}{3}\right)$$



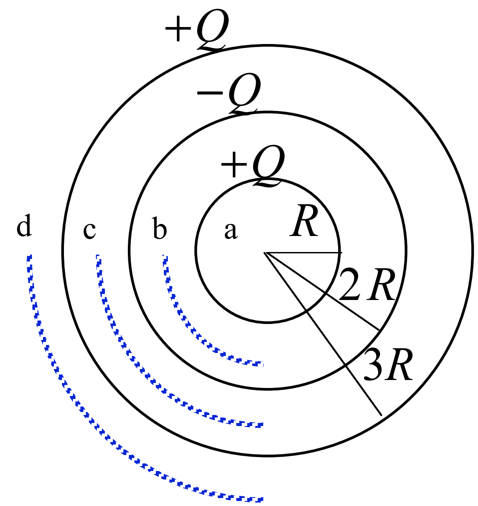
$$= \underline{-\frac{2x^3}{3} + x^2y + 2xy^2 - \frac{y^3}{3} + C}$$

$$\nabla\cdot\mathbf{E}(\mathbf{x}) = 4x - 2y - 4x + 2y = 0$$

Problem 3.40

Concentric spherical shells, radius R, 2R, 3R, charges +Q, -Q, +Q resp. Determine the electric field at all radii.

Gauss's law $\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{q_{encl}}{\varepsilon_0}$



$$(r < R): E = 0$$

$$(R < r < 2R): E4\pi r^2 = +Q/\varepsilon_0; \quad E = \underline{Q \big/ 4\pi\varepsilon_0 r^2}$$

$$(2R < r < 3R): E = 0$$

$$(3R < r): E4\pi r^2 = +Q/\varepsilon_0; \quad E = \underline{Q \big/ 4\pi\varepsilon_0 r^2}$$

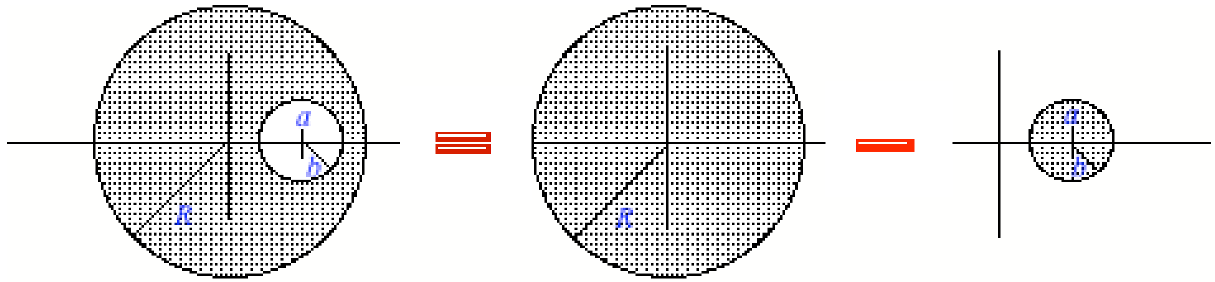
$$(r_b = 1.5R): E_b = Q \big/ 4\pi\varepsilon_0 r_b^2$$

$$(r_d = 3.5R): E_d = Q \big/ 4\pi\varepsilon_0 r_d^2$$

$$\frac{E_d}{E_b} = \left(\frac{r_b}{r_d}\right)^2 = \left(\frac{3}{7}\right)^2 = \frac{9}{49}$$

Problem 3.42

Solid sphere with uniform charge density has a smaller sphere hollowed out. Find field in the cavity.



Find field inside a large (L) sphere radius R with a uniform charge density, Use Gauss's law in spherical coordinates -> Convert to Cartesian coordinates.

$$\oint_S \mathbf{E}\cdot d\mathbf{A} = \frac{q_{encl}}{\varepsilon_0}$$

$$E_r 4\pi r^2 = \frac{\rho V_r}{\varepsilon_0} = \frac{Q}{V_R}\frac{V_r}{\varepsilon_0} = \frac{Q}{\varepsilon_0}\frac{r^3}{R^3}$$

$$\mathbf{E}_{\text{Large}} = \frac{Q}{4\pi\varepsilon_0}\frac{r}{R^3}\hat{\mathbf{r}} = \frac{Q}{4\pi\varepsilon_0}\frac{\left(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}\right)}{R^3}$$

Find field inside a small sphere, radius b, with a uniform charge density, offset in the x direction by a distance a < R-b. Use Cartesian coordinates

$$\mathbf{E}_{\text{Large}} - \mathbf{E}_{\text{Small}} = \frac{Q}{4\pi\varepsilon_0}\frac{\left(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}\right)}{R^3} - \frac{q}{4\pi\varepsilon_0}\left[\frac{(x-a)\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}}{b^3}\right]$$

$$= \frac{Q}{4\pi\varepsilon_0}\frac{\left(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}\right)}{R^3} - \frac{Qb^3}{4\pi\varepsilon_0 R^3}\left[\frac{(x-a)\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}}{b^3}\right]$$

$$q = \rho V_b = \frac{QV_b}{V_R} = \frac{Qb^3}{R^3}$$

$$= \underline{\frac{Qa}{4\pi\varepsilon_0 R^3}}\hat{\mathbf{i}} = \underline{\frac{\rho a}{3\varepsilon_0}}\hat{\mathbf{i}}$$

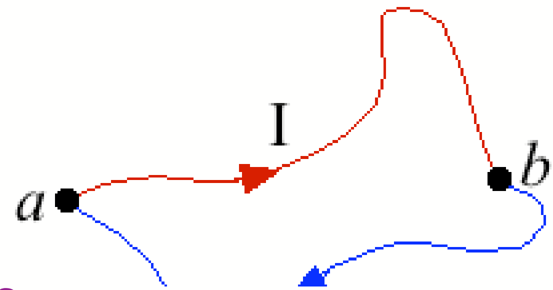
Electric potential differences

Stokes's theorem: $\oint_C \mathbf{E}\cdot d\ell = \oint_S (\nabla\times\mathbf{E})\cdot\hat{\mathbf{n}}\,dA$ but $\nabla\times\mathbf{E}=0$

Integral over closed path: $\oint_C \mathbf{E}\cdot d\ell = 0$

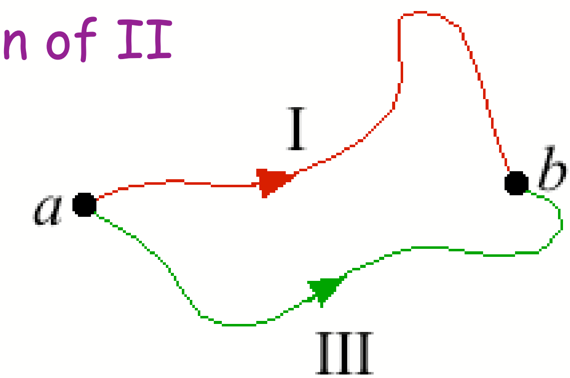
Integral around loop

$$\int_a^b \mathbf{E}\cdot d\ell_I + \int_b^a \mathbf{E}\cdot d\ell_{II} = 0$$



Reverse direction of II

$$\int_a^b \mathbf{E}\cdot d\ell_I = -\int_b^a \mathbf{E}\cdot d\ell_{II} = \int_a^b \mathbf{E}\cdot d\ell_{III}$$



Integral is independent of path

Integral depends only on end points

Let V(x) be "potential" at x

$$V(\mathbf{x}_b) - V(\mathbf{x}_a) = -\int_a^b \mathbf{E}\cdot d\ell$$

Integral is work done (by an external agent) in moving a unit charge from a to b.