
PHY481: Electromagnetism

Potential $V(\mathbf{x})$

Potential energy U

Potential energy density $u(\mathbf{x})$

Electric potential differences

Stokes's theorem: $\oint_C \mathbf{E} \cdot d\ell = \oint_S (\nabla \times \mathbf{E}) \cdot \hat{\mathbf{n}} dA$ but $\nabla \times \mathbf{E} = 0$

Integral over closed path: $\oint_C \mathbf{E} \cdot d\ell = 0$

Integral around loop

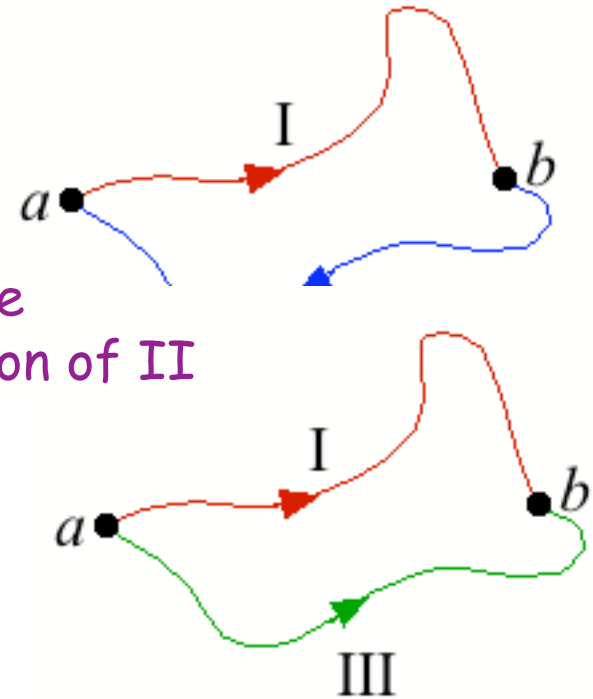
$$\int_a^b \mathbf{E} \cdot d\ell_I + \int_b^a \mathbf{E} \cdot d\ell_{II} = 0$$

$$\int_a^b \mathbf{E} \cdot d\ell_I = -\int_b^a \mathbf{E} \cdot d\ell_{II} = \int_a^b \mathbf{E} \cdot d\ell_{III}$$

Integral is independent of path

Integral depends only on end points

Reverse direction of II



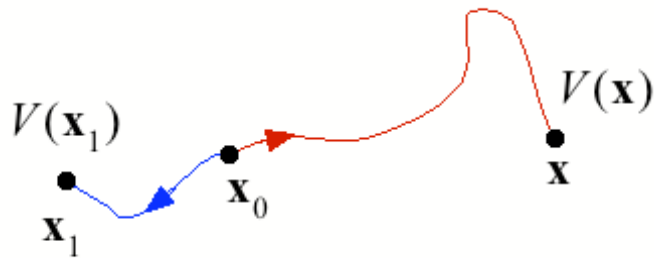
Let $V(x)$ be
"potential" at x

$$V(\mathbf{x}_b) - V(\mathbf{x}_a) = -\int_a^b \mathbf{E} \cdot d\ell$$

Integral is work done (by an external agent) in moving a unit charge from a to b .

V defined up to an additive constant

$V(\mathbf{x})$ and $V(\mathbf{x}_1)$ are potentials relative to potential at point $\mathbf{x}_0$



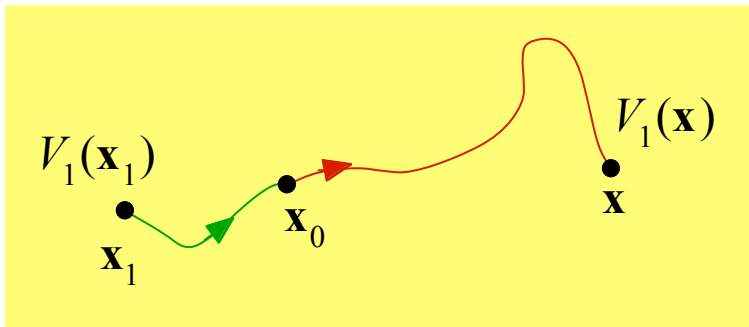
$$V(\mathbf{x}) = - \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{E} \cdot d\ell$$

$\mathbf{x}_0 \rightarrow \mathbf{x}$

$$V(\mathbf{x}_1) = - \int_{\mathbf{x}_0}^{\mathbf{x}_1} \mathbf{E} \cdot d\ell$$

$\mathbf{x}_0 \rightarrow \mathbf{x}_1$

Find $V_1(\mathbf{x})$: potential at point $\mathbf{x}$ relative to potential at point $\mathbf{x}_1$



Changing reference point results in a constant added to V

$$C = -V(\mathbf{x}_1)$$

$$\begin{aligned} V_1(\mathbf{x}) &= - \int_{\mathbf{x}_1}^{\mathbf{x}} \mathbf{E} \cdot d\ell & \mathbf{x}_1 \rightarrow \mathbf{x}_0 \rightarrow \mathbf{x} \\ &= \left(- \int_{\mathbf{x}_1}^{\mathbf{x}_0} \mathbf{E} \cdot d\ell \right) + \left(- \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{E} \cdot d\ell \right) \\ V_1(\mathbf{x}) &= -V(\mathbf{x}_1) + V(\mathbf{x}) \end{aligned}$$

Potential and field of a point charge

- Relationship between E and V

$$\mathbf{E}(\mathbf{x}) = -\nabla V(\mathbf{x})$$

where $V(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}') d^3x'}{|\mathbf{x} - \mathbf{x}'|}$

Charge density for a point charge q at origin

$$\rho(\mathbf{x}') = q\delta^3(\mathbf{x}')$$

- Potential of point charge q at the origin

$$V(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}') d^3x'}{|\mathbf{x} - \mathbf{x}'|} = \frac{q}{4\pi\epsilon_0} \int \frac{\delta^3(\mathbf{x}') d^3x'}{|\mathbf{x} - \mathbf{x}'|}$$

$$V(r) = \frac{q}{4\pi\epsilon_0 |\mathbf{x}|} = \frac{q}{4\pi\epsilon_0 r} \quad |\mathbf{x}| = r$$

- Check field is correct

$$\begin{aligned} \mathbf{E}(r) &= -\nabla V(r) = -\frac{\partial V(r)}{\partial r} \hat{\mathbf{r}} \\ &= -\frac{q}{4\pi\epsilon_0} \frac{\partial}{\partial r} \left[\frac{1}{r} \right] \hat{\mathbf{r}} = \frac{q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}} \end{aligned} \quad \text{OK!}$$

Potential due to spherical **shell** of charge

- Spherical shell, charge Q , radius R , at distance z (or r)

$$V(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}') d^3x'}{|\mathbf{x} - \mathbf{x}'|}$$

$$\rho(\mathbf{x}') d^3x' = \sigma R^2 \sin\theta d\theta d\phi$$

$$\mathbf{x}' = R \cos\theta \hat{\mathbf{k}} + R \sin\theta \hat{\mathbf{j}}$$

$$\mathbf{x} - \mathbf{x}' = (z - R \cos\theta) \hat{\mathbf{k}} - R \sin\theta \hat{\mathbf{j}}$$

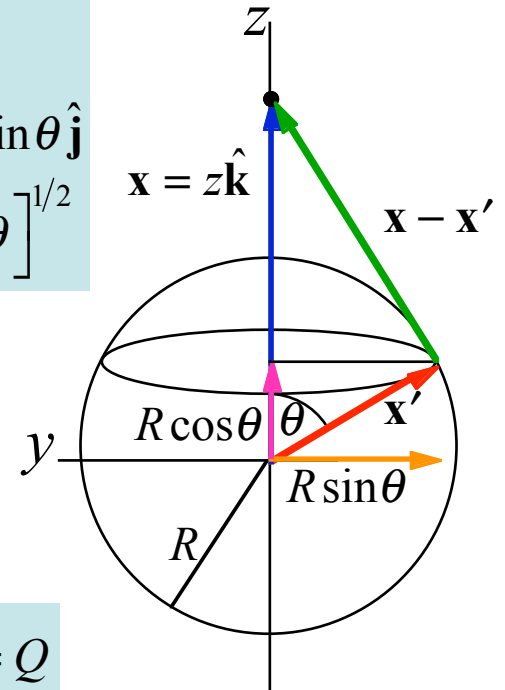
$$|\mathbf{x} - \mathbf{x}'| = [z^2 + R^2 - 2zR \cos\theta]^{1/2}$$

$$V(z) = \frac{\sigma R^2}{4\pi\epsilon_0} \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta [z^2 + R^2 - 2zR \cos\theta]^{-1/2}$$

integrals.wolfram.com

$$= \frac{2\pi\sigma R}{4\pi\epsilon_0 z} [(z + R) - (\pm)(z - R)] \begin{cases} (+) & z > R \\ (-) & z < R \end{cases}$$

$$4\pi R^2 \sigma = Q$$



Point charge

Potential
Outside

$$V(z) = \frac{Q}{4\pi\epsilon_0 z}, \quad z > R$$

Constant

Potential
Inside

$$V(z) = \frac{Q}{4\pi\epsilon_0 R}, \quad z < R$$

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ENTER ANY FUNCTION choose a random example >>

$\int \frac{\text{Sin}[x]}{(a^2 + b^2 + 2 a b \text{Cos}[x])^{1/2}} dx$

COMPUTE

TraditionalForm StandardForm InputForm OutputForm

$$\int \frac{\sin(x)}{\sqrt{a^2 + 2 b \cos(x) a + b^2}} dx =$$
$$-\frac{\sqrt{a^2 + 2 b \cos(x) a + b^2}}{a b}$$

Time to calculate: 0.05 second

Potential for sphere with uniform charge density

- Sphere, uniform density ρ , radius R , at distance z (or r)

$$V(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}') d^3x'}{|\mathbf{x} - \mathbf{x}'|}$$

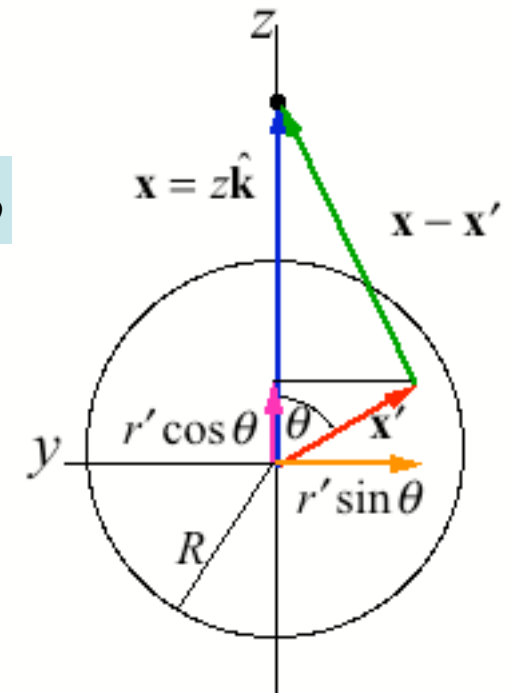
$$\mathbf{x}' = r' \cos\theta \hat{\mathbf{k}} + r' \sin\theta \hat{\mathbf{j}}$$

$$\mathbf{x} - \mathbf{x}' = (z - r' \cos\theta) \hat{\mathbf{k}} - r' \sin\theta \hat{\mathbf{j}}$$

$$\rho(\mathbf{x}') d^3x' = \rho r'^2 dr' \sin\theta d\theta d\phi$$

$$V(z) = \frac{2\pi\rho}{4\pi\epsilon_0} \int_0^R r'^2 dr' \int_0^\pi \sin\theta d\theta \left[z^2 + r'^2 - 2zr' \cos\theta \right]^{-1/2}$$

$$= \frac{2\pi\rho}{4\pi\epsilon_0 z} \int_0^R r' dr' \left\{ \left[(z + r')^2 \right]^{1/2} - \left[(z - r')^2 \right]^{1/2} \right\}$$



- Outside the sphere $z > R$

$$4\pi R^3 \rho / 3 = Q$$

$$V(z) = \frac{2\pi\rho}{4\pi\epsilon_0 z} \int_0^R r' dr' \{ (z + r') - (z - r') \} = \frac{4\pi\rho}{4\pi\epsilon_0 z} \left[\frac{r'^3}{3} \right]_0^R$$

$$= \frac{Q}{4\pi\epsilon_0 z}$$

Uniformly charged sphere (cont'd)

- Inside the sphere

$$V(z) = \frac{2\pi\rho}{4\pi\epsilon_0 z} \int_0^R r' dr' \left\{ \left[(z + r')^2 \right]^{1/2} - \left[(z - r')^2 \right]^{1/2} \right\}$$

$$= \frac{2\pi\rho}{4\pi\epsilon_0 z} \int_0^R r' dr' \left\{ (z + r') - (\pm)(z - r') \right\} \text{ Need positive root}$$

Split integral into two parts $0 < r' < z$ & $z < r' < R$

$r' < z$ use +

$r' > z$ use -

$$\begin{aligned} V(z) &= \frac{4\pi\rho}{4\pi\epsilon_0 z} \left\{ \int_0^z r'^2 dr' + \int_z^R zr' dr' \right\} \\ &= \frac{4\pi\rho}{4\pi\epsilon_0 z} \left\{ \frac{z^3}{3} + \frac{zR^2}{2} - \frac{z^3}{2} \right\} = \frac{4\pi\rho}{8\pi\epsilon_0} \left\{ R^2 - \frac{z^2}{3} \right\} \end{aligned}$$

$$4\pi R^3 \rho / 3 = Q$$

$$= \frac{Q}{8\pi\epsilon_0 R} \left\{ 3 - \frac{z^2}{R^2} \right\}$$

Uniformly charged sphere (cont'd)

■ $V \rightarrow E$

Outside

$$V(r) = \frac{Q}{4\pi\epsilon_0 r}$$

$$\mathbf{E}(r) = -\nabla V(r) = -\frac{Q}{4\pi\epsilon_0} \frac{\partial}{\partial r} \left[\frac{1}{r} \right] \hat{\mathbf{r}} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}$$

$$= \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}$$

Inside

$$V(r) = \frac{Q}{8\pi\epsilon_0 R} \left\{ 3 - \frac{r^2}{R^2} \right\}$$

$$\mathbf{E}(r) = -\nabla V(r) = \frac{Q}{4\pi\epsilon_0} \frac{r}{R^3} \hat{\mathbf{r}}$$

■ $E \rightarrow V$

Outside

$$V(r) = -\int_{\infty}^r \mathbf{E} \cdot d\ell = -\frac{Q}{4\pi\epsilon_0} \int_{\infty}^r \frac{dr'}{r'^2}$$

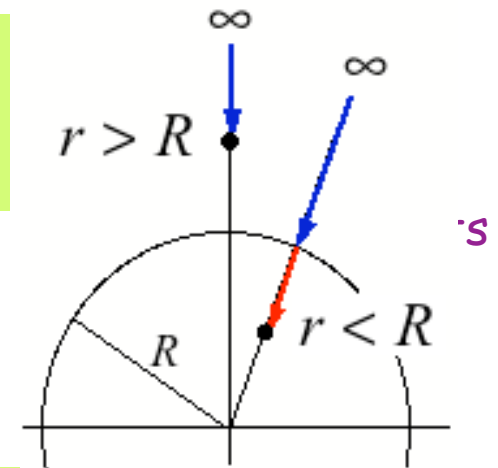
$$= \frac{Q}{4\pi\epsilon_0 r}$$

Inside

$$V(r) = -\frac{Q}{4\pi\epsilon_0} \left\{ \int_{\infty}^R \frac{dr'}{r'^2} + \int_R^r \frac{r' dr'}{R^3} \right\}$$

$$= \frac{Q}{4\pi\epsilon_0} \left\{ \frac{1}{R} - \frac{r^2}{2R^3} + \frac{1}{2R} \right\}$$

$$= \frac{Q}{8\pi\epsilon_0 R} \left\{ 3 - \frac{r^2}{R^2} \right\}$$



Potential **energy** for point charges

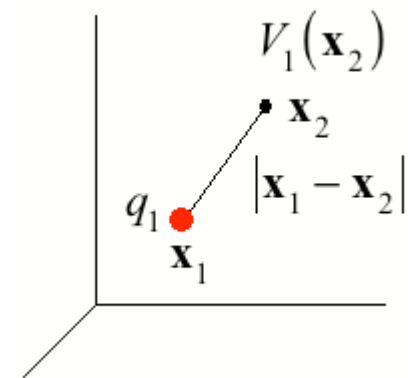
- Electric potential (should be a - sign in front of integral)

$$V(\mathbf{x}) = -\int_{\infty}^{\mathbf{x}} \mathbf{E} \cdot d\ell = W$$

Work done (by an external agent) moving a **unit charge** from ∞ to $\mathbf{x}$

$$V_1(\mathbf{x}_2) = \frac{q_1}{4\pi\epsilon_0 |\mathbf{x}_1 - \mathbf{x}_2|}$$

Potential at $\mathbf{x}_2$ due to charge q_1 at point $\mathbf{x}_1$



- Potential energy

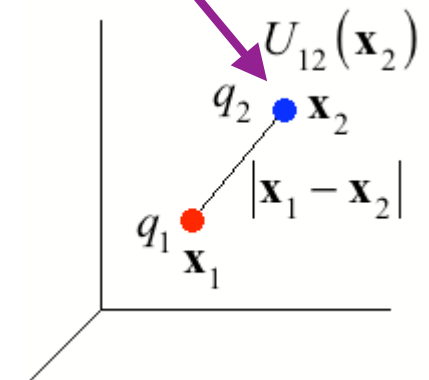
$$U(\mathbf{x}) = qV(\mathbf{x})$$

Potential energy U of charge q .

$$U_{12}(\mathbf{x}_2) = q_2 V_1(\mathbf{x}_2)$$

Potential energy U_{12} of the charge q_2 due the potential V_1

Put charge q_2 at point $\mathbf{x}_2$.



Potential **energy** for charge distributions

- Potential energy for multiple point charges

$$U = \frac{1}{2} \sum_{i \neq j} q_i V_j(\mathbf{x}_i, \mathbf{x}_j) = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{4\pi\epsilon_0 |\mathbf{x}_i - \mathbf{x}_j|}$$

$i \neq j$, do not count charge with itself.
1/2 corrects for $q_i q_j$ and $q_j q_i$

- Generalize for charge distributions

$$U = \frac{1}{2} \int \rho(\mathbf{x}) d^3x \int \frac{\rho(\mathbf{x}')}{4\pi\epsilon_0 |\mathbf{x} - \mathbf{x}'|} d^3x'$$

$$V(\mathbf{x}) = \int \frac{\rho(\mathbf{x}')}{4\pi\epsilon_0 |\mathbf{x} - \mathbf{x}'|} d^3x'$$

$$U = \frac{1}{2} \int \rho(\mathbf{x}) V(\mathbf{x}) d^3x$$

Charge produces potential and the same charge gains potential energy?

- Where is this energy?

The charge distribution produces an electric field. We interpret the energy as being stored by (in) the field!

Energy of an electric field

- Express U as a function of $\mathbf{E}$ only !

$$U = \frac{1}{2} \int \rho V d^3x \quad \nabla \cdot \mathbf{E} = \rho / \epsilon_0$$

Identity

$$U = \frac{\epsilon_0}{2} \int (\nabla \cdot \mathbf{E}) V d^3x \quad \nabla \cdot (V \mathbf{E}) = V \nabla \cdot \mathbf{E} + \mathbf{E} \cdot (\nabla V)$$

$$= \frac{\epsilon_0}{2} \int \nabla \cdot (V \mathbf{E}) d^3x - \frac{\epsilon_0}{2} \int \mathbf{E} \cdot (\nabla V) d^3x \quad \mathbf{E} = -\nabla V$$

Gauss's theorem

$$= \frac{\epsilon_0}{2} \int \nabla \cdot (V \mathbf{E}) d^3x + \frac{\epsilon_0}{2} \int E^2 d^3x$$

$$\oint_{Vol} \nabla \cdot (V \mathbf{E}) d^3x = \oint_S V \mathbf{E} \cdot d\mathbf{A}$$

$$= \frac{\epsilon_0}{2} \oint_S V \mathbf{E} \cdot d\mathbf{A} + \frac{\epsilon_0}{2} \oint_{Vol} E^2 d^3x$$

$$\xrightarrow{S, Vol \rightarrow \infty} U = \frac{\epsilon_0}{2} \oint_{all} E^2 d^3x$$

Field energy

Energy density

$$u_E(\mathbf{x}) = \epsilon_0 E^2(\mathbf{x}) / 2$$