
PHY481: Electromagnetism

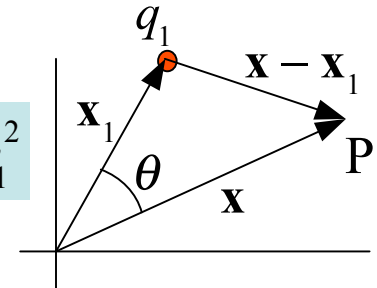
Multipoles
HW 3b hints

Approximating $1/|\mathbf{x}-\mathbf{x}'|$

Potential of single charge

$$V(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \frac{q_1}{|\mathbf{x} - \mathbf{x}_1|}$$

$$|\mathbf{x}| = r; \quad |\mathbf{x}_1| = r_1; \quad |\mathbf{x} - \mathbf{x}_1|^2 = r^2 - 2rr_1 \cos\theta_1 + r_1^2$$



$$\frac{1}{|\mathbf{x} - \mathbf{x}_1|} = \frac{1}{\sqrt{r^2 - 2rr_1 \cos\theta_1 + r_1^2}} = \frac{1}{r\sqrt{1 - 2\frac{r_1}{r} \cos\theta_1 + \frac{r_1^2}{r^2}}} = \frac{1}{r\sqrt{1 - \epsilon}}$$

$$\frac{1}{\sqrt{1 - \epsilon}} = \left(1 + \frac{\epsilon}{2} + \frac{3\epsilon^2}{8} + \dots\right); \quad \epsilon = 2\frac{r_1}{r} \cos\theta_1 - \frac{r_1^2}{r^2}; \quad \epsilon^2 = 4\frac{r_1^2}{r^2} \cos^2\theta_1 - 4\frac{r_1^3}{r^3} \cos\theta_1 + \frac{r_1^4}{r^4}$$

$$\frac{1}{|\mathbf{x} - \mathbf{x}_1|} = \frac{1}{r} \left(1 + \frac{\epsilon}{2} + \frac{3\epsilon^2}{8} + \dots\right) = \frac{1}{r} \left(1 + \frac{r_1}{r} \cos\theta_1 - \frac{r_1^2}{2r^2} + \frac{3}{8} \left(4\frac{r_1^2}{r^2} \cos^2\theta_1 - 4\frac{r_1^3}{r^3} \cos\theta_1 + \frac{r_1^4}{r^4}\right) + \dots\right)$$

Approximating $1/|\mathbf{x}-\mathbf{x}'|$

Potential of two charges

$$V(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q_1}{|\mathbf{x} - \mathbf{x}_1|} + \frac{q_2}{|\mathbf{x} - \mathbf{x}_2|} \right)$$

$$\frac{q_1}{|\mathbf{x} - \mathbf{x}_1|} + \frac{q_2}{|\mathbf{x} - \mathbf{x}_2|} = \frac{Q}{r} + \frac{q_1 r_1 \cos \theta_1 + q_2 r_2 \cos \theta_2}{r^2} + \dots$$

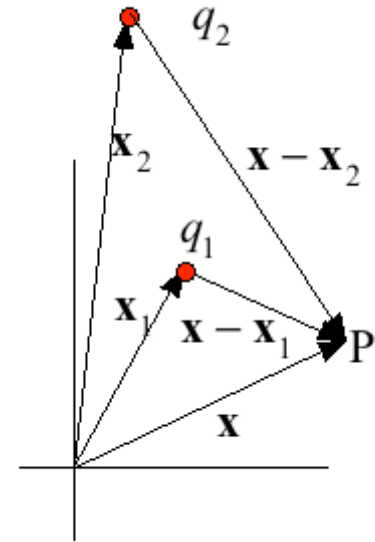
$$V(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \left(\frac{Q}{r} + \frac{\hat{\mathbf{r}} \cdot \mathbf{p}}{r^2} + \frac{\hat{\mathbf{r}} \cdot \mathbf{Q}_2 \cdot \hat{\mathbf{r}}}{r^3} + \dots \right)$$

dipole moment

$$\mathbf{p} = q_1 \mathbf{x}_1 + q_2 \mathbf{x}_2$$

quadrupole moment

$$\mathbf{Q}_2 = \frac{q_1}{2} (3\mathbf{x}_1 \mathbf{x}_1 - r_1^2 \mathbf{I}) + \frac{q_2}{2} (3\mathbf{x}_2 \mathbf{x}_2 - r_2^2 \mathbf{I})$$



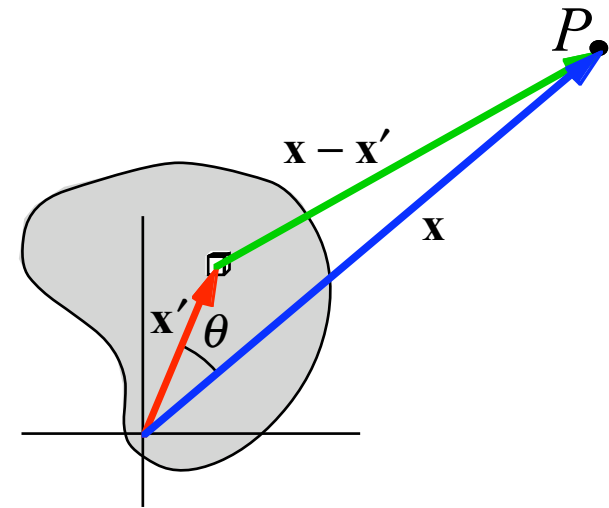
Multipole expansion of V

$$V(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}') d^3x'}{|\mathbf{x} - \mathbf{x}'|}$$

$$V(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \left[\frac{1}{r} \int \rho(\mathbf{x}') d^3x' + \right.$$

$$+ \frac{1}{r^2} \int r' \cos\theta \rho(\mathbf{x}') d^3x'$$

$$+ \frac{1}{r^3} \int r'^2 \left(\frac{3}{2} \cos^2\theta - \frac{1}{2} \right) \rho(\mathbf{x}') d^3x'$$



higher-pole terms

+ ...]

$$V(\mathbf{x}) = \frac{1}{4\pi\epsilon_0 r} \sum_{n=0}^{\infty} \int (r'/r)^n P_n(\cos\theta) \rho(\mathbf{x}') d^3x'$$

Legendre
polynomials

$$\underline{P_0(\cos\theta) = 1; P_1(\cos\theta) = \cos\theta; P_2(\cos\theta) = \frac{3}{2} \cos^2\theta - \frac{1}{2}}$$

Potential of a finite dipole

- Charges $+q$ and $-q$ separated by d along the z -axis

Delta function
charge density

$$\rho(\mathbf{r}') = [q\delta(\mathbf{r}' + \hat{\mathbf{k}}d/2) - q\delta(\mathbf{r}' - \hat{\mathbf{k}}d/2)]$$

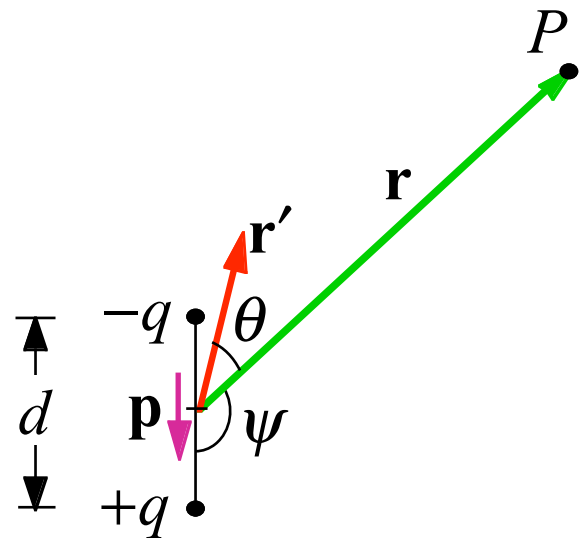
monopole term

$$\int \rho(\mathbf{r}') d^3x' = 0$$

dipole term

$$r' \cos \theta = \hat{\mathbf{r}} \cdot \mathbf{r}'$$

$$\begin{aligned} \int r' \cos \theta \rho(\mathbf{r}') d^3x' &= \int \hat{\mathbf{r}} \cdot \mathbf{r}' \rho(\mathbf{r}') d^3x' \\ &= \hat{\mathbf{r}} \cdot \int \mathbf{r}' \rho(\mathbf{r}') d^3x' \\ &= \hat{\mathbf{r}} \cdot (-qd \hat{\mathbf{k}}) = \hat{\mathbf{r}} \cdot \mathbf{p} \end{aligned}$$



Angle ψ is between $\mathbf{p}$ and $\mathbf{r}$

$$V(\mathbf{x}) \approx \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{4\pi\epsilon_0 r^2}$$

with

$$\mathbf{p} = -p\hat{\mathbf{k}}, \quad p = qd$$

dipole moment

$$V(\mathbf{x}) \approx \frac{p \cos \psi}{4\pi\epsilon_0 r^2}$$

Higher poles $\rightarrow 0$
 $\sim r^{-4}$ or faster

Point dipole

- Potential of a point dipole $\mathbf{p}$ (units are C·m)

Limit of finite dipole as
 $d \rightarrow 0$, with qd constant

$$p = \lim_{d \rightarrow 0} (qd) \Big|_{qd \text{ constant}}$$

Exact

$$V(\mathbf{x}) = \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{4\pi\epsilon_0 r^2}$$

with $\mathbf{p} = p \hat{\mathbf{k}}$

$$V(r, \theta) = \frac{p \cos \theta}{4\pi\epsilon_0 r^2}$$

- Electric field of point dipole $\mathbf{E}(\mathbf{x}) = -\nabla V(\mathbf{x})$

$$\begin{aligned} \mathbf{E}(\mathbf{x}) &= \frac{-1}{4\pi\epsilon_0} \nabla \left(\frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} \right) = \frac{-1}{4\pi\epsilon_0} \nabla \left(\frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \right) \\ &= \frac{-1}{4\pi\epsilon_0} \left[\mathbf{p} \cdot \mathbf{r} \nabla \left(\frac{1}{r^3} \right) + \left(\frac{1}{r^3} \right) \nabla (\mathbf{p} \cdot \mathbf{r}) \right] \end{aligned}$$

$$\begin{aligned} \nabla (\mathbf{p} \cdot \mathbf{r}) &= \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i} (p_j x_j) \\ &= \hat{\mathbf{e}}_i p_i = \mathbf{p} \end{aligned}$$

$$\begin{aligned} &= \frac{-1}{4\pi\epsilon_0} \left[\mathbf{p} \cdot \mathbf{r} \left(\frac{-3}{r^4} \hat{\mathbf{r}} \right) + \left(\frac{\mathbf{p}}{r^3} \right) \right] \\ &= \frac{1}{4\pi\epsilon_0 r^3} [3(\mathbf{p} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{p}] \end{aligned}$$

Torque and energy of dipole in electric field

- Torque on electric dipole

$$\mathbf{N} = \mathbf{p} \times \mathbf{E} = pE \sin \theta (-\hat{\mathbf{k}})$$

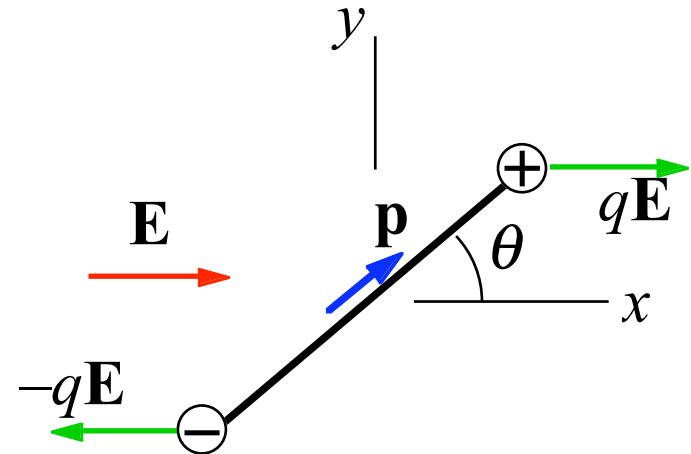
- Energy of electric dipole

$$U = -\mathbf{p} \cdot \mathbf{E} = -pE \cos \theta$$

- Force on electric dipole

$$\begin{aligned} \mathbf{F}(\mathbf{x}) &= -\nabla U(\mathbf{x}) = \nabla (\mathbf{p} \cdot \mathbf{E}(\mathbf{x})) \\ &= (\mathbf{p} \cdot \nabla) \mathbf{E}(\mathbf{x}) \end{aligned}$$

$$F_j(\mathbf{x}) = p_i \frac{\partial E_j(\mathbf{x})}{\partial x_i}$$



If $\mathbf{E}(\mathbf{x})$ is constant, $\mathbf{F}(\mathbf{x}) = 0$

Problems 3.22, 23, 25

3.22 Potential inside and outside a spherical shell, uniform charge density
(See lecture 11)

3.23 Potential and field inside and outside a sphere, charge density $\rho = C r$
(See lecture 11 for one technique, or **use Gauss's law for $\mathbf{E}$** , then $\mathbf{E} \rightarrow V$)

3.25 Laplace's equation in a **charge free** region

Poisson's equation

$$-\nabla^2 V(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\epsilon_0}$$

Laplace's equation

$$\nabla^2 V(\mathbf{x}) = 0$$

Problem 3.26, 30

3.26 Self energy of a sphere with charge Q and radius R .

Two ways to get the energy:

1) Build up charge

Do the work necessary
to assemble the charge

2) Surface integral

Self energy calculation

Both should give the same answer, that diverges as $R \rightarrow 0$

3.30 Multipole expansion for the potential on z -axis for a ring of charge.

Taylor series expansion
of $f(x)$ about $x = 0$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \dots$$
$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2$$

Problem 3.32

a) Point dipole force on a displaced charge

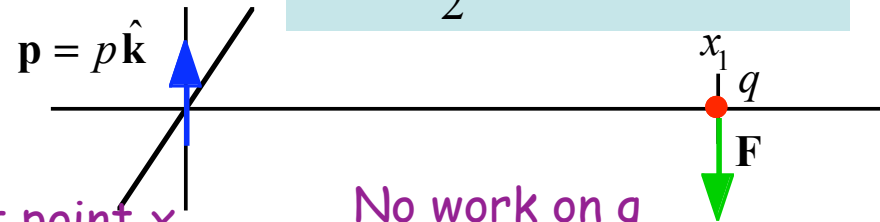
Field of dipole $\mathbf{p}$ (in +z direction)

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3} (2\cos\theta \hat{\mathbf{r}} + \sin\theta \hat{\boldsymbol{\theta}})$$

Field at point x_1

Force on q at point x_1

$$\text{at } \theta = \frac{\pi}{2}, \cos\theta = 0, \hat{\boldsymbol{\theta}} = -\hat{\mathbf{k}}$$



No work on q

$$\mathbf{E} = \frac{-p\hat{\mathbf{k}}}{4\pi\epsilon_0 x_1^3}$$

$$\mathbf{F} = q\mathbf{E}$$

$$W$$

$\mathbf{F}$ on dipole = $-\mathbf{F}$ on charge

Check this by direct calculation

Consider a bar between the point charge and the dipole

Will it rotate when released?