PHY481: Electromagnetism

HW 3b Solutions

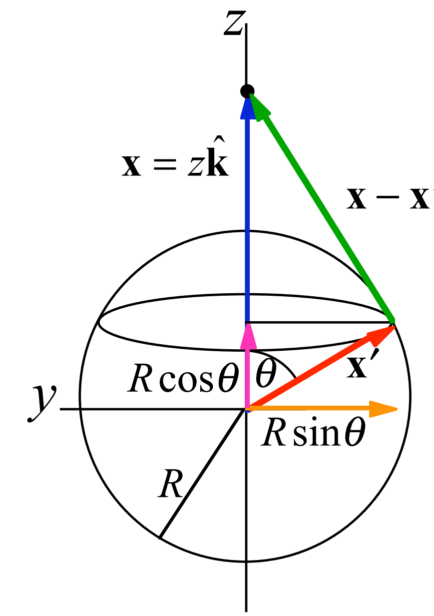
Practice Exam Hints

HW 3 Problem 3.22

Potential inside and outside a spherical shell, uniform charge density
(See lecture 11)

$$V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

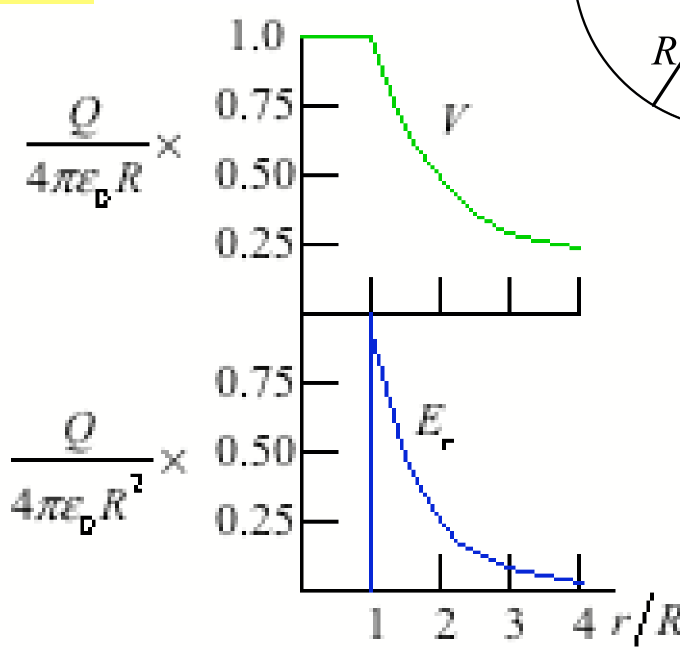
$$4\pi R^2\sigma = Q$$



$$V(z) = \frac{\sigma R^2}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \left[z^2 + R^2 - 2zR\cos\theta\right]^{-1/2}$$

$$V(z) = \frac{2\pi\sigma R}{4\pi\varepsilon_0 z}[(R+z)-(\pm)(R-z)] \quad \begin{matrix}(-) & z>R \\ (+) & z<R\end{matrix}$$

$$V(z) = \underline{\frac{Q}{4\pi\varepsilon_0 R}},\ z<R \qquad V(z) = \underline{\frac{Q}{4\pi\varepsilon_0 z}},\ z>R$$



$$\mathbf{E}(z) = \underline{-\nabla V = 0},\quad r<R$$

$$E_r(r) = \underline{Q\big/4\pi\varepsilon_0 r^2},\quad r>R$$

HW 3 Problem 3.23

a) Potential and field inside and outside a sphere, charge density $\rho = C\,r$
(See lecture 11 for one technique, or use Gauss's law for **E**, then **E** --> V)

Gauss's law $\oint_S \mathbf{E}\cdot d\mathbf{A} = q/\varepsilon_0$

$$Q = \int \rho(\mathbf{x}')\,d^3x' = 4\pi\int_0^a Cr'^3 dr' = \pi C a^4; \quad C = \frac{Q}{\pi a^4}$$

Inside sphere (r < a)

$$4\pi r^2 E = \frac{4\pi}{\varepsilon_0}\int_0^r Cr'^3 dr' = \frac{\pi}{\varepsilon_0}Cr^4 \qquad \underline{E_r = Cr^2/4\varepsilon_0}$$

Outside sphere (r > a)

$$4\pi r^2 E = Q/\varepsilon_0 = \pi C a^4/\varepsilon_0 \qquad \underline{E_r = Ca^4/4\varepsilon_0 r^2}$$

E --> V $\quad V(r) = -\int_\infty^r \mathbf{E}\cdot d\ell$

Inside sphere

$$V(r) = -\frac{Ca^4}{4\varepsilon_0}\int_\infty^a r'^{-2}dr' - \frac{C}{4\varepsilon_0}\int_a^r r'^2 dr' = \frac{C}{12\varepsilon_0}\left[4a^3 - r^3\right]$$

Outside sphere

$$V(r) = -\frac{Ca^4}{4\varepsilon_0}\int_\infty^r r'^{-2}dr' = \underline{\frac{Ca^4}{4\varepsilon_0 r}}$$

b) Add a surface charge density σ_0 to the sphere

Inside sphere

$$\underline{V(r) = \frac{\sigma_0 R}{\varepsilon_0} + \frac{C}{12\varepsilon_0}\left[4a^3 - r^3\right]} \qquad E_r = \underline{\frac{Cr^2}{4\varepsilon_0}}$$

Outside sphere

$$V(r) = \underline{\frac{\sigma_0 a^2}{\varepsilon_0 r} + \frac{Ca^4}{4\varepsilon_0 r}} \qquad E_r = \underline{\frac{\sigma_0 a^2}{\varepsilon_0 r^2} + \frac{Ca^4}{4\varepsilon_0 r^2}}$$

HW 3 Problem 3.25

Laplace's equation in a charge free region

Poisson's equation $-\nabla^2 V(\mathbf{x}) = \dfrac{\rho(\mathbf{x})}{\varepsilon_0}$

In charge free regions Laplace's equation $\nabla^2 V(\mathbf{x}) = 0$

$$V(r,\theta) = r^{-3}(C_1 \cos^2\theta + C_2)$$

$$\frac{\partial V}{\partial r} = -3r^{-4}(C_1 \cos^2\theta + C_2)$$

$$\frac{\partial V}{\partial \theta} = -2r^{-3} C_1 \cos\theta \sin\theta$$

$$\nabla^2 V(r,\theta) = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right] + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial V}{\partial \theta}\right]$$

$$\nabla^2 V(r,\theta) = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left[-3r^{-2}(C_1\cos^2\theta + C_2)\right] + \frac{-2C_1}{r^5 \sin\theta}\frac{\partial}{\partial\theta}\left[\cos\theta \sin^2\theta\right]$$

$$0 = \frac{6}{r^5}(C_1 \cos^2\theta + C_2) - \frac{2C_1}{r^5}\left[-1 + 3\cos^2\theta\right]$$

$$C_1 = -3C_2$$

HW 3 Problem 3.26

Self energy of a sphere with charge Q and radius R.

Two ways to get the energy:

1) Build up charge $U = \int dU = \int V dq$ Do the work necessary to assemble the charge

$$U = \int V dq = \frac{1}{4\pi\varepsilon_0 R}\int_0^Q q\,dq = \underline{\frac{Q^2}{8\pi\varepsilon_0 R}}$$

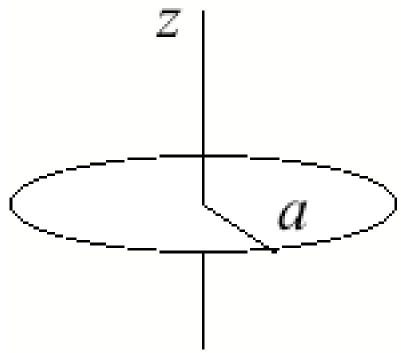
2) Surface integral $U = \frac{1}{2}\int \rho(\mathbf{x}) V(\mathbf{x})\, d^3x'$ Self energy calculation

$$U = \frac{1}{2} Q \frac{Q}{4\pi\varepsilon_0 R} = \underline{\frac{Q^2}{8\pi\varepsilon_0 R}}$$

Both give the same answer, that diverges as R -> 0

HW 3 Problem 3.30

Multipole expansion for the potential on z-axis for a ring of charge.



Taylor series expansion of *f(x)* about *x* = 0

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \ldots$$

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2$$

$$V(0,0,z) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\left(a^2+z^2\right)^{1/2}} = \frac{Q}{4\pi\varepsilon_0 z}\left(1+\frac{a^2}{z^2}\right)^{-1/2}$$

$$= \frac{Q}{4\pi\varepsilon_0 z}\left(1 - \frac{a^2}{2z^2} + \frac{3a^4}{8z^4} + \ldots\right)$$

monopole

quadrupole

hexadecapole

No dipole term as there is a complete symmetry in z for the charge distribution

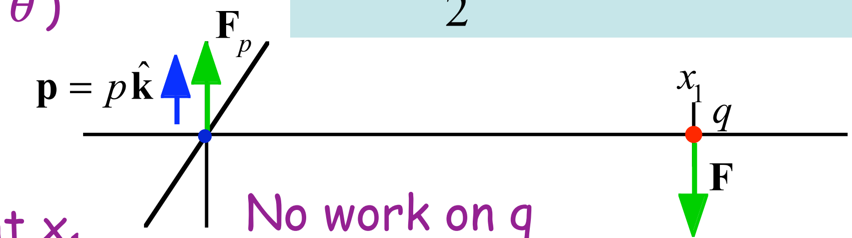
HW 3 Problem 3.32

a) Point dipole force on a displaced charge

$$\text{at } \theta = \frac{\pi}{2},\ \cos\theta = 0, \hat{\boldsymbol{\theta}} = -\hat{\mathbf{k}}$$

Field of dipole **p** (in +z direction,vs. r & θ)

$$\mathbf{E} = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$



Field at point x_1

$$\mathbf{E} = \frac{-p\,\hat{\mathbf{k}}}{4\pi\varepsilon_0 x_1^3}$$

Force on q at point x_1

$$\mathbf{F} = q\mathbf{E} = \frac{-qp\,\hat{\mathbf{k}}}{4\pi\varepsilon_0 r^3}$$

No work on q

$$W = \int_{\infty}^{x_1} \mathbf{F}\cdot\hat{\mathbf{i}}\,dx = 0$$

Newton's 3rd law

F on dipole = -**F** on charge

$$\mathbf{F}_p = \frac{+qp\,\hat{\mathbf{k}}}{4\pi\varepsilon_0 x_1^3}$$

Check by direct calculation

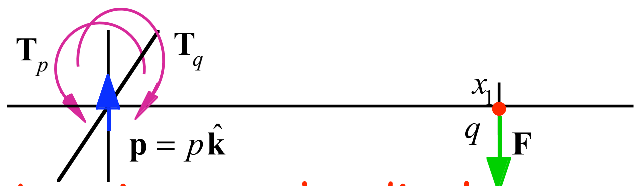
$$F_z = (\mathbf{p}\cdot\nabla)E_z = p_z\frac{\partial}{\partial z}E_z = \frac{qp}{4\pi\varepsilon_0}\frac{\partial}{\partial z}\left(\frac{z}{\left[(x-x_1)^2 + y^2 + z^2\right]^{3/2}}\right)$$

$$F_z\Big|_{x,y,z=0} = \frac{qp}{4\pi\varepsilon_0}\left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right)\Bigg|_{x,y,z=0} = \underline{\frac{qp}{4\pi\varepsilon_0 x_1^3}}$$

Forces are "balanced", but what about torque?
Will a bar between the point charge and the dipole rotate if released?

Violates angular momentum conservation: No external torques -> **L** is constant

HW 3 Problem 3.32 (cont'd)



Use pivot at the dipole

they balance

Torque from the charge $\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F} = \hat{\mathbf{j}}\, qp\big/4\pi\varepsilon_0 x_1^2$

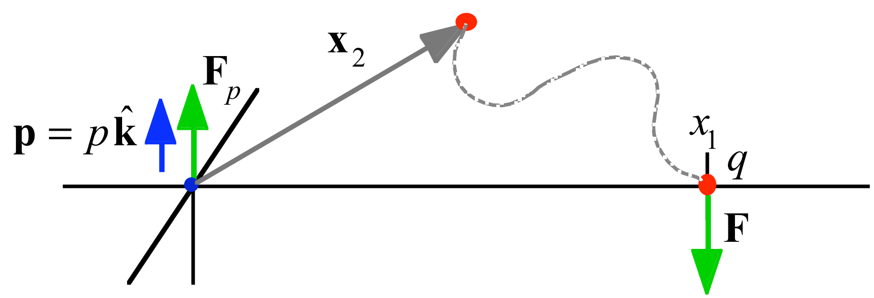
Torque from the dipole $\boldsymbol{\tau} = \mathbf{p}\times\mathbf{E} = -\hat{\mathbf{j}}\, qp\big/4\pi\varepsilon_0 x_1^2$

b) Force on q at point $\mathbf{x}_2$, and work needed to move charge there.

Work needed = Potential energy change

$$\Delta U = qV(\mathbf{x}_2) - qV(\mathbf{x}_1); \quad V(\mathbf{x}_1) = 0$$

$$U(\mathbf{x}_2) = q\frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\varepsilon_0 r^3} = \quad q\frac{pz_2}{4\pi\varepsilon_0 r_2^3}$$



F = q **E** at the point $\mathbf{x}_2$

$$\mathbf{F} = q\mathbf{E} = \frac{qp}{4\pi\varepsilon_0 r_2^3}\left(2\cos\theta_2\,\hat{\mathbf{r}} + \sin\theta_2\,\hat{\boldsymbol{\theta}}\right) = \quad \frac{qp}{4\pi\varepsilon_0 r_2^4}\left(2z_2\,\hat{\mathbf{r}} + \left(x_2^2 + y_2^2\right)^{1/2}\hat{\boldsymbol{\theta}}\right)$$

$$\mathbf{F} = \frac{qp}{4\pi\varepsilon_0 r_2^3}\left(3\cos\theta_2\,\hat{\mathbf{r}} - \hat{\mathbf{k}}\right) \quad = \frac{qp}{4\pi\varepsilon_0 r_2^3}\left(3\frac{z_2}{r_2}\hat{\mathbf{r}} - \hat{\mathbf{k}}\right)$$

using $\hat{\mathbf{k}} = \cos\theta_2\hat{\mathbf{r}} - \sin\theta_2\,\hat{\boldsymbol{\theta}}$

HW 3 Problem 3.33

Dipole field and potential in spherical coordinates

$$\mathbf{p} = p_1\,\hat{\mathbf{i}} + p_2\,\hat{\mathbf{j}} + p_3\,\hat{\mathbf{k}} \qquad \mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$$

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$V = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\varepsilon_0 r^3} = \frac{1}{4\pi\varepsilon_0 r^2}\left(p_1\frac{x}{r} + p_2\frac{y}{r} + p_3\frac{z}{r}\right)$$

$$V = \frac{1}{4\pi\varepsilon_0 r^2}\left(p_1\sin\theta\cos\phi + p_2\sin\theta\sin\phi + p_3\cos\theta\right)$$

$$\mathbf{E} = -\nabla V = -\left\{\frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\boldsymbol{\phi}}\right\}$$

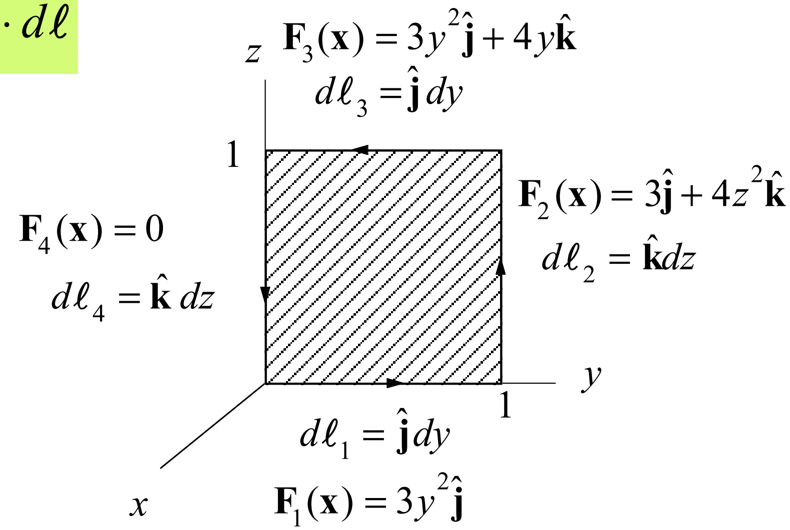
$$\begin{aligned}\mathbf{E} = \frac{1}{4\pi\varepsilon_0 r^3}&\left\{2\left(p_1\sin\theta\cos\phi + p_2\sin\theta\sin\phi + p_3\cos\theta\right)\hat{\mathbf{r}}\right.\\ &+\left(-p_1\cos\theta\cos\phi - p_2\cos\theta\sin\phi + p_3\sin\theta\right)\hat{\boldsymbol{\theta}} +\\ &\left.+\left(p_1\sin\phi - p_2\cos\phi\right)\hat{\boldsymbol{\phi}}\right\}\end{aligned}$$

Practice problems

1) Stokes's theorem: $\oint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dA = \oint_C \mathbf{F} \cdot d\ell$

$$\mathbf{F}(\mathbf{x}) = (2xy + 3y^2)\,\hat{\mathbf{j}} + (4yz^2)\,\hat{\mathbf{k}}$$

$$\oint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = \oint_C \mathbf{F} \cdot d\ell = \underline{4/3}$$



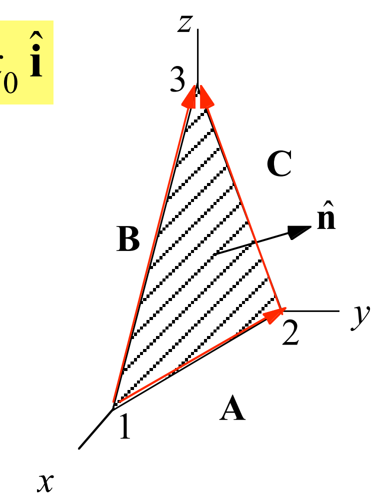
2) Gauss's law

a) Cross product of 2 vectors in the plane $\mathbf{A} = y_0\,\hat{\mathbf{j}} - x_0\,\hat{\mathbf{i}}$

$$\hat{\mathbf{n}} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|}$$

b) Dot product of normal with vector in the plane

$$\hat{\mathbf{n}} \cdot (\mathbf{x} - \mathbf{x}') = 0 \qquad \mathbf{x} = \left(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}\right)$$



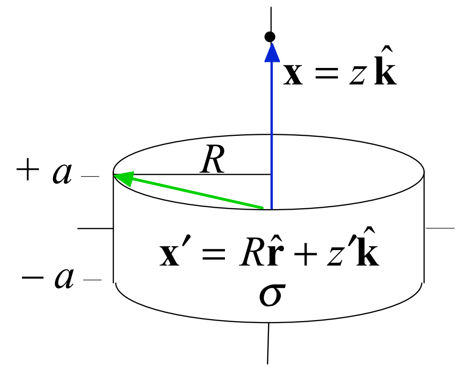
Practice problems

3a) Gauss's law: $\oint_S \mathbf{E}\cdot d\mathbf{A} = q_{encl}/\varepsilon_0$ $\qquad E_{plane} = \dfrac{\sigma}{2\varepsilon_0}$

4) Electric field for ring of charge

$$E = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')\,d^3x'$$

$$E_z = \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi} R\,d\phi' \int_{-a}^{+a} dz' \frac{(z-z')}{\left(R^2+(z-z')^2\right)^{3/2}}$$



b) Approximation for $z \gg a,R$ Write as function of $\dfrac{a}{z};\dfrac{R}{z}$

5) Prove $\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$

$$\mathbf{C} = \mathbf{A}\times\mathbf{B}$$
$$C_k = \varepsilon_{klm}A_l B_m$$

Practice Problems

6) Point dipole potential

$$\mathbf{p} = p_0 \hat{\mathbf{k}} \qquad V(r,\theta) = \frac{p_0 \cos\theta}{4\pi\varepsilon_0 r^2}$$

$$\mathbf{E} = -\nabla V = -\left(\hat{\mathbf{r}} \frac{\partial V}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}} \frac{1}{r \sin\theta} \frac{\partial V}{\partial \phi} \right)$$

7) Dirac delta function

$$\int d^3x' \, f(\mathbf{x}') \delta^3(\mathbf{x} - \mathbf{x}') = f(\mathbf{x})$$

a) (i) $\displaystyle\int_2^6 dx \, (3x^2 - 2x - 1)\delta(x-3) = (3 \cdot 9 - 2 \cdot 3 - 1)$