Exam 1 Solutions
Charge on a conductor
Exam 1

1) $V(x), E(x), \rho(x)$

a) $V(x) \rightarrow E(x)$

$$-\nabla V = E$$

b) $V(x) \rightarrow \rho(x)$

$$-\nabla^2 V = \frac{\rho}{\varepsilon_0}$$

c) $E(x) \rightarrow \rho(x)$

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

d) $E(x) \rightarrow V(x)$

$$-\int E \cdot dl = V$$

e) $\rho(x) \rightarrow E(x)$

$$\frac{1}{4\pi\varepsilon_0} \int \frac{\rho(x')(x-x')}{|x-x'|^3} d^3x' = E(x)$$

f) $\rho(x) \rightarrow V(x)$

$$\frac{1}{4\pi\varepsilon_0} \int \frac{\rho(x')}{|x-x'|} d^3x' = V(x)$$
Exam 1 (cont’d)

2) Prove:
\[
\nabla \times \nabla \times \mathbf{F} = \nabla (\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}
\]

let \((\nabla \times \mathbf{F})_k = G_k = \varepsilon_{k\ell m} \frac{\partial F_m}{\partial x_\ell}\)

\[
(\nabla \times \nabla \times \mathbf{F})_i = \varepsilon_{ijk} \frac{\partial G_k}{\partial x_j} = \varepsilon_{ijk} \varepsilon_{k\ell m} \frac{\partial}{\partial x_j} \frac{\partial F_m}{\partial x_\ell} \\
= (\delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell}) \frac{\partial}{\partial x_j} \frac{\partial F_m}{\partial x_\ell} \\
= \frac{\partial}{\partial x_j} \frac{\partial F_j}{\partial x_i} - \frac{\partial}{\partial x_j} \frac{\partial F_i}{\partial x_j} = \frac{\partial}{\partial x_i} \left( \frac{\partial F_j}{\partial x_j} \right) - \frac{\partial^2}{\partial x_j^2} (F_i) \\
= \nabla (\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}
\]
3 Decompose unit vector in z-direction, \( \hat{k} \), into components along the unit vectors in the \( \hat{r} \) and \( \hat{\theta} \) directions.

\[ \hat{k} = \hat{r}\cos\theta - \hat{\theta}\sin\theta \]

\[ \theta = \pi / 2 : \hat{k} = -\hat{\theta} \], with \( r \) in the x-y plane, \( \hat{\theta} \) is downward, \( -\hat{\theta} \) is upward like \( \hat{k} \).

\[ \theta = \pi : \hat{k} = -\hat{r} \], with \( r \) pointing downward, \( -\hat{r} \) points upward like \( \hat{k} \).
Exam 1 (cont'd)

4a) \[ E(x) = \frac{1}{4\pi\epsilon_0} \int \rho(x) \left( \frac{x - x'}{|x - x'|^3} \right) d^3x' \]
\[ = \frac{\sigma}{4\pi\epsilon_0} \int_0^a r'dr' \int_{-\pi/2}^{\pi/2} d\phi \frac{(z\hat{k} - ir'\cos\phi - jr'\sin\phi)}{(z^2 + r'^2)^{3/2}} \]
\[ = \frac{\sigma z\hat{k}}{4\epsilon_0} \int_0^a dr' \frac{r'}{(z^2 + r'^2)^{3/2}} - \frac{\sigma i}{2\pi\epsilon_0} \int_0^a \frac{r'^2dr'}{(z^2 + r'^2)^{3/2}} \]

\[ E_z(z) = \frac{\sigma z}{4\epsilon_0} \int_{z^2}^{a^2 + z^2} \frac{dx}{x^{3/2}} = -\frac{\sigma z}{4\epsilon_0} \left( \frac{1}{\sqrt{z^2}} \right)^{3/2} \]
\[ = \frac{\sigma z}{4\epsilon_0} \left[ \frac{1}{z} - \frac{1}{(z^2 + a^2)^{1/2}} \right] = \frac{\sigma}{4\epsilon_0} \left( 1 - \frac{1}{(1 + a^2/z^2)^{1/2}} \right) \]

b) \[ E_z(z) = \frac{\sigma}{4\epsilon_0} \left( \frac{a^2}{2z^2} + \ldots \right) \approx \frac{\sigma a^2 / 2}{4\epsilon_0 z^2} = \frac{\sigma a^2 / 2}{4\pi\epsilon_0 z^2} = \frac{Q}{4\pi\epsilon_0 z^2} \text{ for } z^2 >> a^2 \]

= Electric field of a point charge $Q$ at the origin.
Exam 1 (cont’d)

5) Two shells, outer shell grounded [ V(b)=0]
   a) Electric field  By Gauss’s Law

   \[ 4\pi r^2 E_r = Q/\varepsilon_0; \quad E_r = \frac{Q}{4\pi \varepsilon_0 r^2} \]

   b) Electric potential

   \[ V(r) = -\int E \cdot d\ell = -\int_{r_b}^{r_a} E_r dr' = -\frac{Q}{4\pi \varepsilon_0} \int_{r_b}^{r_a} \frac{dr'}{r'^2} = \frac{Q}{4\pi \varepsilon_0} \left[ \frac{1}{r} \right]_{r_b}^{r_a} = \frac{Q}{4\pi \varepsilon_0} \left[ \frac{1}{r} - \frac{1}{b} \right] \]

   c) Stored energy from charge and potential

   \[ U = \frac{1}{2} \left[ (+Q)V(a) + (-Q)V(b) \right] = \frac{Q^2}{8\pi \varepsilon_0} \left[ \frac{1}{a} - \frac{1}{b} \right] = \frac{Q^2}{8\pi \varepsilon_0} \frac{(b-a)}{ab} \]

   d) Stored energy from field

   \[ U = \frac{\varepsilon_0}{2} \int E^2(x')d^3x' = \frac{\varepsilon_0}{2} \int_{a}^{b} \left( \frac{Q}{4\pi \varepsilon_0 r'^2} \right)^2 r'^2 dr' \int_{0}^{\pi} \sin \theta d\theta \int_{0}^{2\pi} d\phi \]

   \[ = \frac{Q^2}{8\pi \varepsilon_0} \int_{a}^{b} \frac{dr'}{r'^2} = \frac{Q^2}{8\pi \varepsilon_0} \left[ -\frac{1}{r'} \right]_{a}^{b} = \frac{Q^2}{8\pi \varepsilon_0} \frac{(b-a)}{ab} \]

Note: V(b) = 0
Exam 1 (cont’d)

6a) Dipole moment

\[ p = \int x' \rho(x') d^3 x' \]
\[ d^3 x' = R^2 \sin \theta d\theta d\phi \]

\[ x' = r' = \hat{k}R \cos \theta + \hat{i}R \sin \theta \cos \phi + \hat{j}R \sin \theta \sin \phi \]

\[ Q = 2\pi R^2 (+\sigma), \ 0 < \theta < \pi/2; \quad Q = 2\pi R^2 (-\sigma), \ \pi/2 < \theta < \pi \]

\[ p = \int x' \rho(x') d^3 x' = R^3 \int_{0}^{2\pi} d\phi \left( \int_{0}^{\pi/2} \sigma \cos \theta \sin \theta d\theta + \int_{0}^{\pi/2} -\sigma \cos \theta \sin \theta d\theta \right) \hat{k} + 0\hat{i} + 0\hat{j} \]

\[ = 2\pi R^3 \sigma \left( \left[ \frac{x^2}{2} \right]_{0}^{1} - \left[ \frac{x^2}{2} \right]_{0}^{0} \right) \hat{k} = 2\pi R^3 \sigma \hat{k} = QR \hat{k} \]

b) Point dipole at origin with same dipole moment

\[ p = \sum q_i x_i = (Q) \left( \frac{R}{2} \hat{k} \right) + (-Q) \left( -\frac{R}{2} \hat{k} \right) = QR \hat{k} \]
Conductors in static equilibrium

Why these statements are true?

- Electric field $E$ inside a conductor is zero.
- Potential $V$ inside a conductor is a constant.
- Inside a conductor the charge density $\rho$ is zero.
- Charge $Q$ on a conductor resides only on the surface.
- A net charge $Q_{net}$ here is always paired with charge $-Q_{net}$ elsewhere.
- On a conductor either $V$ or $Q_{net}$, but not both, can be specified.
- At a conductor’s surface, field component $E_t = 0$, component $E_n = \sigma/\varepsilon_0$, but in force calculations, $E_n = \sigma/2\varepsilon_0$ due to only “distant” charges.
- In an empty cavity in a conductor, $E = 0$ and surface $\sigma_{cavity} = 0$.
- The potential $V$ of a conductor can be set by a battery $V_B$.
- The earth is an “infinite” source of charge at a constant $V$.
- $V$ or $Q_{net}$ of conductors (and $Q_{external}$) determine $V$ everywhere.
Boundary value problems: 1-parallel plates

Parallel plates (find potential, field & surface charge)

- The ground is a constant potential, so let $V_1 = 0$.
- Battery draws electrons from plate 2 and puts them on plate 1. Process “continues” until $V_2 = V_B$.

Potential between the plates

- Between the plates the potential depends only on $x$.

Laplace’s equation:

$$ \nabla^2 V(x) = 0 $$

Boundary conditions:

- $V(0) = 0$; $V(d) = V_B$

General solution (ODE)

$$ V(x) = c_1 x + c_2 $$

Apply boundary conditions

- $V(0) = c_2 = 0$
- $V(d) = c_1 d = V_B$; $c_1 = \frac{V_B}{d}$

Potential

$$ V(x) = V_B \frac{x}{d} $$
Parallel plates (cont’d)

Electric field

\[ E = -\nabla V = -\nabla \left( \frac{V}{x} \right) = -\frac{V}{d} \hat{i} \]

\[ E_n \rightarrow \text{Surface charges} \]

\[ E_{1n} = \frac{\sigma_{1r}}{\varepsilon_0} = -\frac{V}{d} \]

\[ E_{2n} = \frac{\sigma_{2\ell}}{\varepsilon_0} = \frac{V}{d} \]

\[ \sigma_{1r} = -\frac{\varepsilon_0 V}{d} \]

\[ \sigma_{2\ell} = +\frac{\varepsilon_0 V}{d} \]

Surface charge

\[ Q_{2\ell} = -Q_{1r} = \sigma A = \frac{\varepsilon_0 V_B A}{d} \]

Force on each plate

\[ F_2 = Q_{2\ell} (E_{2n} / 2) = \frac{\varepsilon_0 V_B A V_B}{d} = \frac{1}{2} \frac{\sigma_{2\ell}^2}{\varepsilon_0} A \]
Parallel plates (cont’d)

Capacitance

Two ways to know the electric field

\[ E = \frac{\sigma}{\epsilon_0}; \quad E = \frac{V_B}{d} \]

Capacitance is a geometrical factor relating the charge on a conductor and its potential.

\[ Q = \sigma A = \frac{\epsilon_0 A}{d} V_B = CV_B \]
\[ C = \frac{\epsilon_0 A}{d} \]

Energy storage

\[ U = \frac{\epsilon_0}{2} \int E^2 d^3 x' \]
\[ = \frac{\epsilon_0}{2} \frac{V_B^2}{d^2} Ad = \frac{1}{2} \frac{\epsilon_0 A}{d} V_B^2 \]
\[ U = \frac{1}{2} CV_B^2 \]