PHY481: Electromagnetism

Exam 1 Solutions

Charge on a conductor

Exam 1

1) V(**x**) , E(**x**), ρ(**x**)

a) $V(\mathbf{x}) \to \mathbf{E}(\mathbf{x})$

$$\underline{-\nabla V = \mathbf{E}}$$

b) $V(\mathbf{x}) \to \rho(\mathbf{x})$

$$\underline{-\nabla^2 V = \frac{\rho}{\varepsilon_0}}$$

c) $\mathbf{E}(\mathbf{x}) \to \rho(\mathbf{x})$

$$\underline{\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}}$$

d) $\mathbf{E}(\mathbf{x}) \to V(\mathbf{x})$

$$\underline{-\int \mathbf{E} \cdot d\mathbf{l} = V}$$

e) $\rho(\mathbf{x}) \to \mathbf{E}(\mathbf{x})$

$$\underline{\frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} d^3x' = \mathbf{E}(\mathbf{x})}$$

f) $\rho(\mathbf{x}) \to V(\mathbf{x})$

$$\underline{\frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' = V(\mathbf{x})}$$

Exam 1 (cont'd)

2) Prove: $\nabla \times \nabla \times \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$

$$\text{let } (\nabla \times \mathbf{F})_k = \mathbf{G}_k = \varepsilon_{k\ell m} \frac{\partial F_m}{\partial x_\ell}$$

$$(\nabla \times \nabla \times \mathbf{F})_i = \varepsilon_{ijk} \frac{\partial G_k}{\partial x_j} = \varepsilon_{ijk} \varepsilon_{k\ell m} \frac{\partial}{\partial x_j} \frac{\partial F_m}{\partial x_\ell}$$

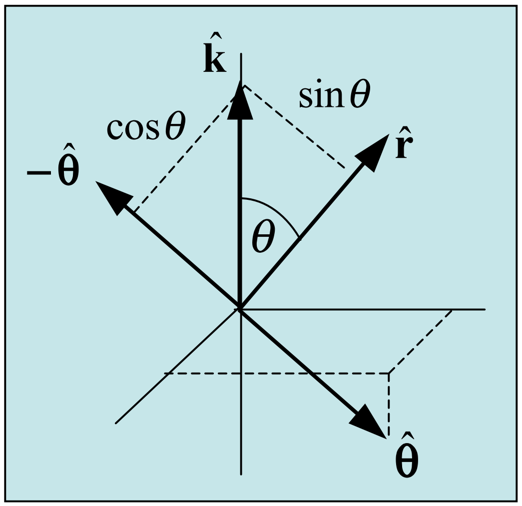
$$= \left(\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}\right) \frac{\partial}{\partial x_j} \frac{\partial F_m}{\partial x_\ell}$$

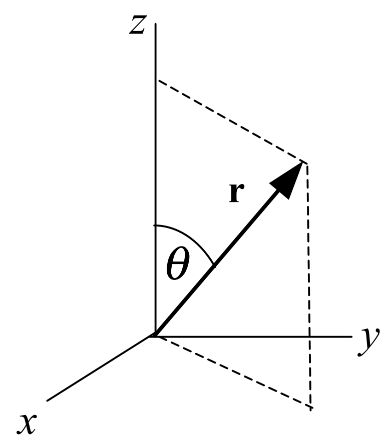
$$= \frac{\partial}{\partial x_j} \frac{\partial F_j}{\partial x_i} - \frac{\partial}{\partial x_j} \frac{\partial F_i}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial F_j}{\partial x_j}\right) - \frac{\partial^2}{\partial x_j^2}(F_i)$$

$$= \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$$

Exam 1 (cont'd)

3 Decompose unit vector in z-direction, $\hat{\mathbf{k}}$, into components along the unit vectors in the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ directions.





$$\underline{\hat{\mathbf{k}} = \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta}$$

$\theta = \pi/2 : \underline{\hat{\mathbf{k}} = -\hat{\boldsymbol{\theta}}}$, with $\mathbf{r}$ in the x-y plane, $\hat{\boldsymbol{\theta}}$ is downward, $-\hat{\boldsymbol{\theta}}$ is upward like $\hat{\mathbf{k}}$.

$\theta = \pi : \underline{\hat{\mathbf{k}} = -\hat{\mathbf{r}}}$, with $\mathbf{r}$ pointing downward, $-\hat{\mathbf{r}}$ points upward like $\hat{\mathbf{k}}$.

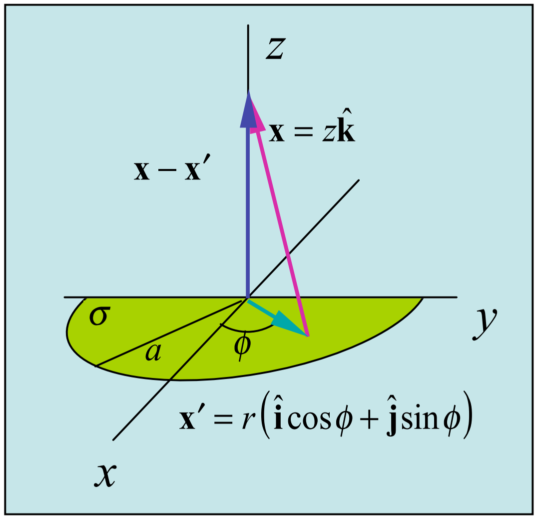
Exam 1 (cont'd)

4a)

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\mathbf{x})\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3x'$$

$$= \frac{\sigma}{4\pi\varepsilon_0}\int_0^a r'dr'\int_{-\pi/2}^{\pi/2} d\phi\frac{\left(z\hat{\mathbf{k}} - \mathbf{i}r'\cos\phi - \hat{\mathbf{j}}r'\sin\phi\right)}{\left(z^2+r'^2\right)^{3/2}}$$

$$= \frac{\sigma z\hat{\mathbf{k}}}{4\varepsilon_0}\int_0^a dr'\frac{r'}{\left(z^2+r'^2\right)^{3/2}} - \frac{\sigma\mathbf{i}}{2\pi\varepsilon_0}\int_0^a\frac{r'^2dr'}{\left(z^2+r'^2\right)^{3/2}}$$



$$E_z(z) = \frac{\sigma z}{4\varepsilon_0}\int_{z^2}^{a^2+z^2}\frac{dx}{x^{3/2}} = -\frac{\sigma z}{4\varepsilon_0}\frac{1}{x^{1/2}}\Bigg|_{z^2}^{a^2+z^2}$$

$$= \frac{\sigma z}{4\varepsilon_0}\left[\frac{1}{z} - \frac{1}{\left(z^2+a^2\right)^{1/2}}\right] = \underline{\frac{\sigma}{4\varepsilon_0}\left(1 - \frac{1}{\left(1+a^2/z^2\right)^{1/2}}\right)}$$

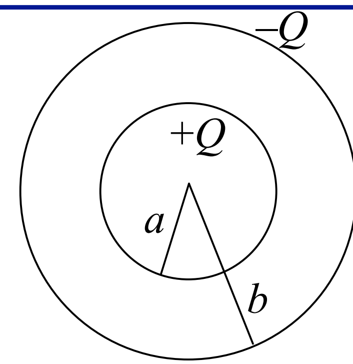
b)

$$E_z(z) = \frac{\sigma}{4\varepsilon_0}\left(\frac{a^2}{2z^2}+\ldots\right) \approx \frac{\sigma a^2/2}{4\varepsilon_0 z^2} = \frac{\sigma\pi a^2/2}{4\pi\varepsilon_0 z^2} = \frac{Q}{4\pi\varepsilon_0 z^2} \quad \text{for } z^2 >> a^2$$

= Electric field of a point charge Q at the origin.

Exam 1 (cont'd)

5) Two shells, outer shell grounded [V(b)=0)]



a) Electric field By Gauss's Law

$$4\pi r^2 E_r = Q/\varepsilon_0 ; \qquad E_r = \frac{Q}{4\pi\varepsilon_0 r^2}$$

b) Electric potential

$$V(r) = -\int_b^r \mathbf{E}\cdot d\ell = -\int_b^r E_r dr' = -\frac{Q}{4\pi\varepsilon_0}\int_b^r \frac{dr'}{r'^2} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r'}\right]_b^r = \underline{\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r}-\frac{1}{b}\right]}$$

c) Stored energy from charge and potential

Note: $V(b) = 0$

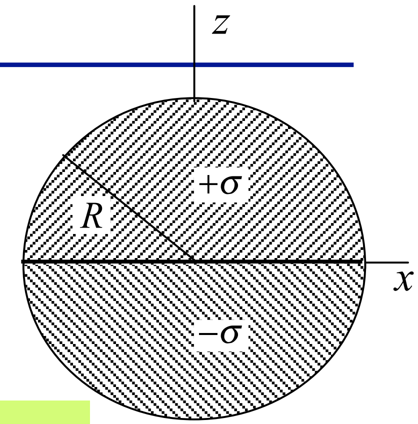
$$U = \frac{1}{2}\Big[\big(+Q\big)V(a) + (-Q)V(b)\Big] = \frac{Q^2}{8\pi\varepsilon_0}\left[\frac{1}{a}-\frac{1}{b}\right] = \frac{Q^2}{8\pi\varepsilon_0}\frac{(b-a)}{ab}$$

d) Stored energy from field

$$U = \frac{\varepsilon_0}{2}\int E^2(x')d^3x' = \frac{\varepsilon_0}{2}\int_a^b\left(\frac{Q}{4\pi\varepsilon_0 r'^2}\right)^2 r'^2 dr'\int_0^\pi \sin\theta d\theta\int_0^{2\pi} d\phi$$

$$= \frac{Q^2}{8\pi\varepsilon_0}\int_a^b\frac{dr'}{r'^2} = \frac{Q^2}{8\pi\varepsilon_0}\left[-\frac{1}{r'}\right]_a^b = \frac{Q^2}{8\pi\varepsilon_0}\frac{(b-a)}{ab}$$

Exam 1 (cont'd)



6a) Dipole moment $\mathbf{p} = \int \mathbf{x}'\rho(\mathbf{x}')d^3x'$ $\quad d^3x' = R^2 \sin\theta d\theta d\phi$

$$\mathbf{x}' = \mathbf{r}' = \hat{\mathbf{k}}R\cos\theta + \hat{\mathbf{i}}R\sin\theta\cos\phi + \hat{\mathbf{j}}R\sin\theta\sin\phi$$

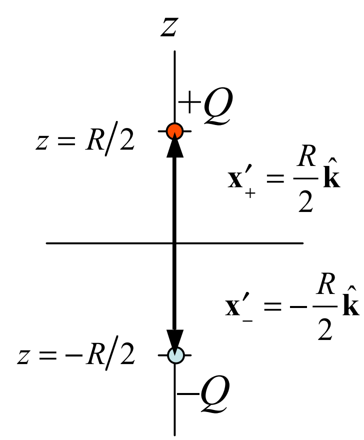
$$Q = 2\pi R^2(+\sigma),\ 0<\theta<\pi/2; \quad Q = 2\pi R^2(-\sigma),\ \pi/2<\theta<\pi$$

$$\mathbf{p} = \int \mathbf{x}'\rho(\mathbf{x}')d^3x' = R^3 \int_0^{2\pi} d\phi \left(\int_0^{\pi/2} \sigma\cos\theta\sin\theta d\theta + \int_{\pi/2}^{\pi} -\sigma\cos\theta\sin\theta d\theta \right)\hat{\mathbf{k}} + 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}}$$

$$= 2\pi R^3\sigma\left(\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^2}{2}\right]_1^0 \right)\hat{\mathbf{k}} = 2\pi R^3\sigma\hat{\mathbf{k}} = QR\hat{\mathbf{k}}$$

b) Point dipole at origin with same dipole moment

$$\mathbf{p} = \sum q_i\mathbf{x}_i = (Q)\left(\frac{R}{2}\hat{\mathbf{k}}\right) + (-Q)\left(-\frac{R}{2}\hat{\mathbf{k}}\right) = QR\hat{\mathbf{k}}$$



Conductors in static equilibrium

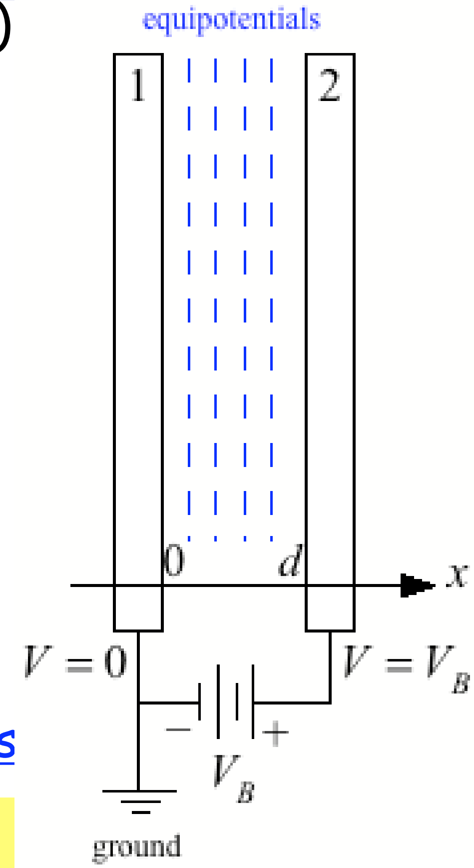
Why these statements are true ?

- Electric field **E** inside a conductor is zero.
- Potential V inside a conductor is a constant.
- Inside a conductor the charge density ρ is zero.
- Charge Q on a conductor resides only on the surface.
- A net charge Q_{net} here is always paired with charge $-Q_{net}$ elsewhere.
- On a conductor either V or Q_{net}, but not both, can be specified.
- At a conductor's surface, field component $E_t = 0$, component $E_n = \sigma/\varepsilon_0$, but in force calculations, $E_n = \sigma/2\varepsilon_0$ due to only "distant" charges.
- In an empty cavity in a conductor, **E** = 0 and surface $\sigma_{cavity} = 0$.
- The potential V of a conductor can be set by a battery V_B.
- The earth is an "infinite" source of charge at a constant V .
- V or Q_{net} of conductors (and $Q_{external}$) determine V everywhere.

Boundary value problems: 1-parallel plates

Parallel plates (find potential, field & surface charge)

The ground is a constant potential, so let $V_1 = 0$.
Battery draws electrons from plate 2 and puts them on plate 1. Process "continues" until $V_2 = V_B$.



Potential between the plates

Between the plates the potential depends only on *x*.

Laplace's equation: $$\nabla^2 V(x) = 0$$

Boundary conditions: $$V(0) = 0; \quad V(d) = V_B$$

General solution (ODE)

$$\nabla^2 V = \frac{d}{dx}\left(\frac{dV}{dx}\right) = 0; \ \frac{dV}{dx} = c_1$$

$$V(x) = \underline{c_1 x + c_2}$$

Apply boundary conditions

$$V(0) = \underline{c_2 = 0}$$

$$V(d) = c_1 d = V_B; \ \underline{c_1 = \frac{V_B}{d}}$$

Potential

$$V(x) = \underline{V_B \frac{x}{d}}$$

Parallel plates (cont'd)

Electric field

Electric field

$$\mathbf{E} = -\nabla V = -\nabla\left(V_B \frac{x}{d}\right) \quad \underline{-\frac{V_B}{d}\hat{\mathbf{i}}}$$

E_n -> Surface charges

$$E_{1n} = \frac{\sigma_{1r}}{\varepsilon_0} = -\frac{V_B}{d}$$

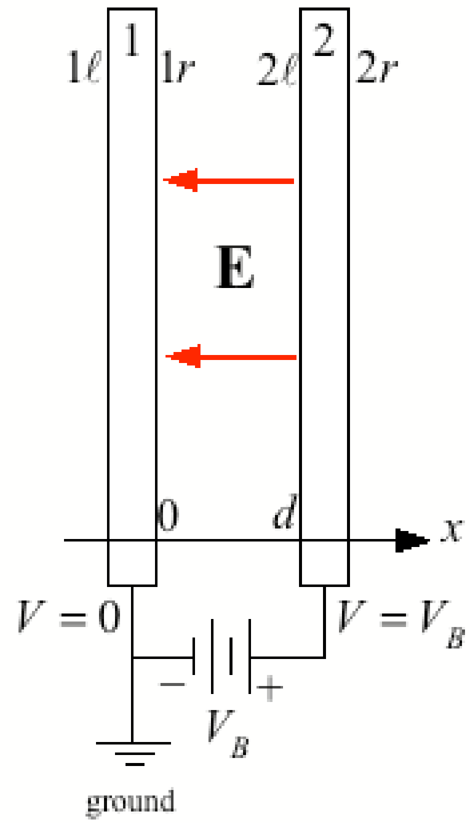
Surface 1*r* charge density

$$\sigma_{1r} = \underline{-\frac{\varepsilon_0 V_B}{d}}$$

$$E_{2n} = \frac{\sigma_{2\ell}}{\varepsilon_0} = \frac{V_B}{d}$$

Surface 2ℓ charge density

$$\sigma_{2\ell} = \underline{+\frac{\varepsilon_0 V_B}{d}}$$



Surface charge

$$Q_{2\ell} = -Q_{1r} = \sigma A = \underline{\frac{\varepsilon_0 V_B A}{d}}$$

Force on each plate

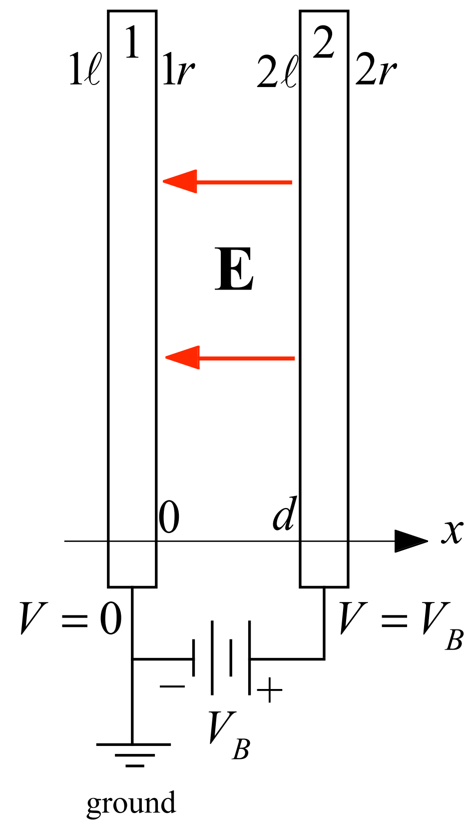
$$F_2 = Q_{2\ell}(E_{2n}/2) = \frac{\varepsilon_0 V_B A}{d}\frac{V_B}{2d} = \underline{\frac{1}{2}\frac{\sigma_{2\ell}^2}{\varepsilon_0}A}$$

Parallel plates (cont'd)

Capacitance

Two ways to know the electric field

$$E = \frac{\sigma}{\varepsilon_0}; \quad E = \frac{V_B}{d}$$



Capacitance is a geometrical factor relating the charge on a conductor and its potential.

$$Q = \sigma A = \frac{\varepsilon_0 A}{d} V_B = C V_B$$

$$C = \frac{\varepsilon_0 A}{d}$$

Energy storage

$$U = \frac{\varepsilon_0}{2} \int E^2 d^3x'$$

$$= \frac{\varepsilon_0}{2} \frac{V_B^2}{d^2} A d = \frac{1}{2} \frac{\varepsilon_0 A}{d} V_B^2$$

$$U = \frac{1}{2} C V_B^2$$