
PHY481: Electromagnetism

Capacitance
Point charge near a conductor
solved by
Method of Images

Parallel plates (cont'd)

Electric field

$$\mathbf{E} = -\nabla V = -\nabla \left(V_B \frac{x}{d} \right)$$

$$\mathbf{E} = -\frac{V_B}{d} \hat{\mathbf{i}}$$

$$E_n = \mathbf{E} \cdot \hat{\mathbf{n}} \quad \text{Normal component}$$

$$E_{1n} = \frac{\sigma_{1r}}{\epsilon_0} = -\frac{V_B}{d}$$

$$\sigma_{1r} = -\frac{\epsilon_0 V_B}{d}$$

Surface 1r charge density

Surface 2ℓ charge density

$$E_{2n} = \frac{\sigma_{2\ell}}{\epsilon_0} = \frac{V_B}{d}$$

$$\sigma_{2\ell} = +\frac{\epsilon_0 V_B}{d}$$

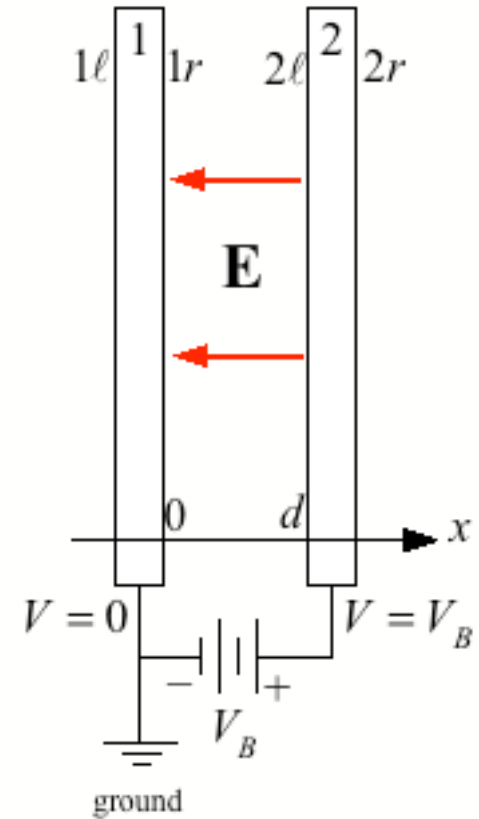
Surface charge

Force on plate 2 due to Q1

$$Q_{2\ell} = -Q_{1r} = \sigma A = \frac{\epsilon_0 V_B A}{d}$$

$$F_2 = Q_{2\ell} (E_{2n} / 2) = \frac{\epsilon_0 V_B A}{d} \frac{V_B}{2d} =$$

$$\frac{1}{2} \frac{\sigma_{2\ell}^2}{\epsilon_0} A$$



Parallel plates (cont'd)

Capacitance

Two ways to know the electric field

$$E = \frac{\sigma}{\epsilon_0}; \quad E = \frac{V_B}{d}$$

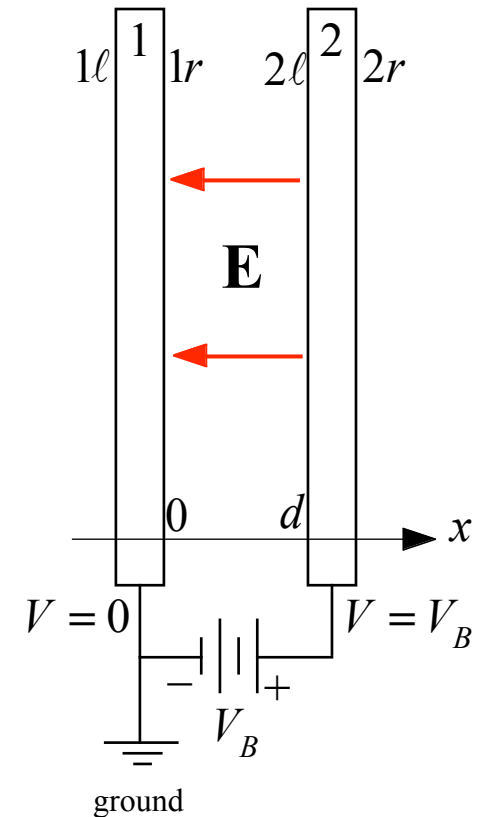
Equate them!

$$\sigma = \frac{\epsilon_0 V_B}{d}$$

Capacitance is a geometrical factor relating the charge on a conductor and its potential.

$$Q = \sigma A = \frac{\epsilon_0 A}{d} V_B = C V_B$$

$$C = \frac{\epsilon_0 A}{d}$$



Energy storage

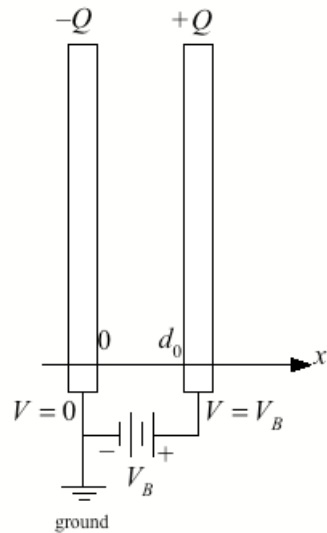
$$U = \frac{\epsilon_0}{2} \int E^2 d^3 x'$$

$$= \frac{\epsilon_0}{2} \frac{V_B^2}{d^2} A d = \frac{1}{2} \frac{\epsilon_0 A}{d} V_B^2$$

$$U = \frac{1}{2} C V_B^2$$

Energy change in separating the plates

Plates charged

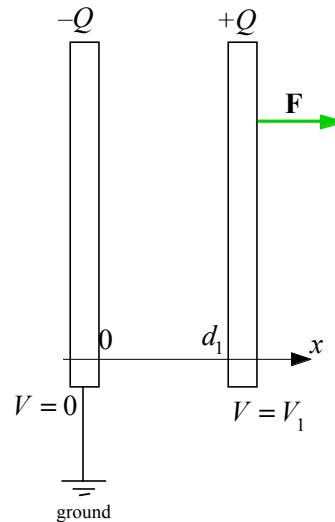


$$Q = CV_B = \epsilon_0 AV_B / d_0$$

Stored energy

$$\mathcal{E} = \frac{1}{2} CV_B^2 = \frac{1}{2} \frac{\epsilon_0 A}{d_0} V_B^2$$

Disconnect battery
and separate plates



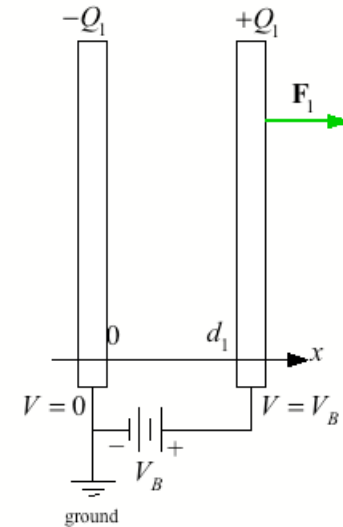
Charge constant at Q
Voltage changes to V_1

$$Q = C_1 V_1 = \epsilon_0 AV_1 / d_1$$

$$V_1 = V_B d_1 / d_0$$

$$\mathcal{E}_1 = \frac{1}{2} C_1 V_1^2 = \frac{d_1}{d_0} \mathcal{E} \quad \uparrow$$

Leave battery connected
and separate plates



Voltage constant at V_B
Charge changes to Q_1

$$Q_1 = C_1 V_B = \epsilon_0 AV_B / d_1$$

$$Q_1 = Q d_0 / d_1$$

$$\mathcal{E}'_1 = \frac{1}{2} C_1 V_B^2 = \frac{d_0}{d_1} \mathcal{E} \quad \downarrow ?$$

Cylindrical capacitor

Capacitance of nested long cylinders

Find charge Q for the given $V = V_B$

Gauss's law between the cylinders

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = 2\pi r L E(r) = \frac{\lambda_a L}{\epsilon_0}$$

Field between the cylinders

$$E(r) = \frac{\lambda_a}{2\pi\epsilon_0 r}$$

Potential between the cylinders

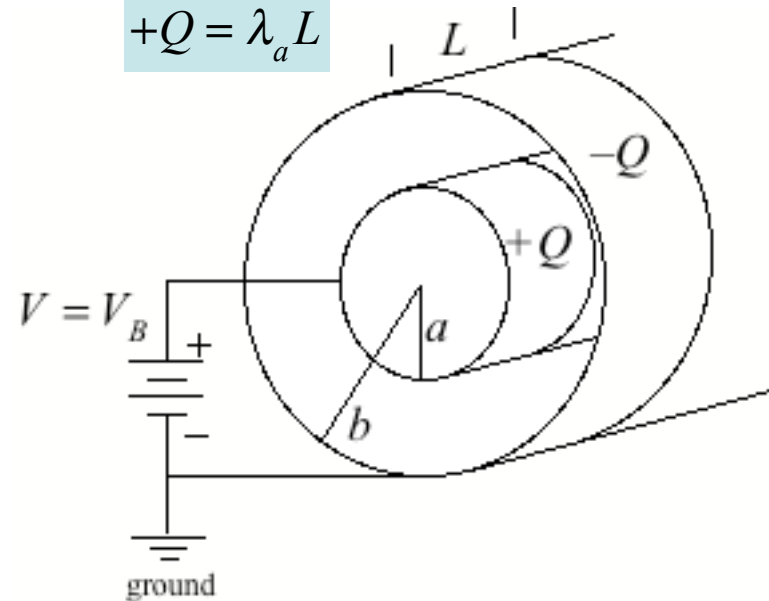
$$V(r) = -\int_b^r E dr = -\frac{\lambda_a}{2\pi\epsilon_0} \int_b^r \frac{dr}{r} = \frac{\lambda_a}{2\pi\epsilon_0} \ln(b/r)$$

Charge and potential relationship

$$V_B = \frac{\lambda_a}{2\pi\epsilon_0} \ln(b/a); \quad \lambda_a = \frac{2\pi\epsilon_0}{\ln(b/a)} V_B$$

$$\lambda_a = \sigma_a (2\pi a)$$

$$+Q = \lambda_a L$$



$$Q = \lambda_a L = \frac{2\pi\epsilon_0 L}{\ln(b/a)} V_B = C V_B$$

Capacitance per unit length

$$C/L = \frac{2\pi\epsilon_0}{\ln(b/a)}$$

Dipole potential & field on the midplane

- Finite dipole potential ($\mathbf{x} \neq \pm z_0 \hat{\mathbf{k}}$)

$$V(\mathbf{x}) = \frac{q}{4\pi\epsilon_0} \left\{ \frac{1}{[x^2 + y^2 + (z - z_0)^2]^{1/2}} - \frac{1}{[x^2 + y^2 + (z + z_0)^2]^{1/2}} \right\}$$

note signs

- Dipole potential on the midplane

$$z = 0, \text{ then } V(x, y, 0) = 0$$

Midplane is an equipotential (0)

- Is Laplace's equation, $\nabla^2 V(\mathbf{x}) = 0$ satisfied everywhere?

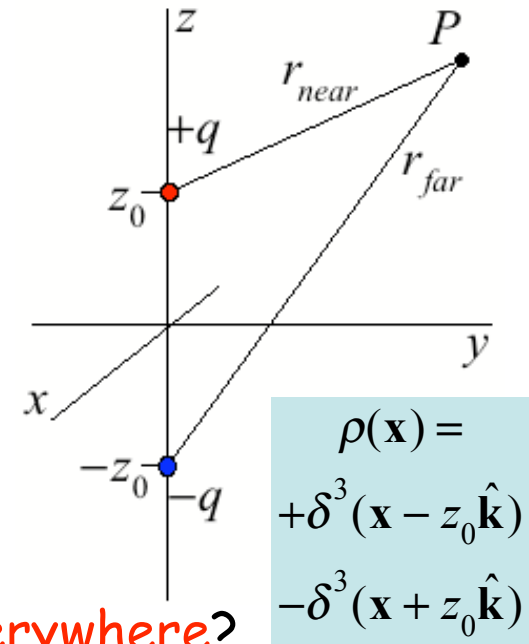
Get dipole field $\mathbf{E}$, then $-\nabla \cdot \mathbf{E} = \nabla \cdot (\nabla V) = \nabla^2 V(\mathbf{x})$

$$\mathbf{E}(x, y, z) = -\nabla V = \frac{q}{4\pi\epsilon_0} \left\{ \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z - z_0)\hat{\mathbf{k}}}{[x^2 + y^2 + (z - z_0)^2]^{3/2}} - \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z + z_0)\hat{\mathbf{k}}}{[x^2 + y^2 + (z + z_0)^2]^{3/2}} \right\}$$

Not obvious that

$$\nabla \cdot \mathbf{E} = 0$$

If it is, then Laplace's Eq.
is satisfied everywhere ($\mathbf{x} \neq \pm z_0 \hat{\mathbf{k}}$)



For a dipole field

$$\nabla^2 V(\mathbf{x}) = -\nabla \cdot \mathbf{E} = 0$$

- $\mathbf{C}(x,y,z)$ is the position dependence of $\mathbf{E}$ -field of the + charge

$$\mathbf{C}(x,y,z) = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z - z_0)\hat{\mathbf{k}}}{[x^2 + y^2 + (z - z_0)^2]^{3/2}} \quad C_x = \frac{x}{[x^2 + y^2 + (z - z_0)^2]^{3/2}}, \text{ etc.}$$

$$\frac{\partial C_x}{\partial x} = \left(\frac{1}{[x^2 + y^2 + (z - z_0)^2]^{3/2}} - \frac{3x^2}{[x^2 + y^2 + (z - z_0)^2]^{5/2}} \right)$$

$$\nabla \cdot \mathbf{C}(x,y,z) = \frac{\partial C_i}{\partial x_i} = \left(\frac{3}{[x^2 + y^2 + (z - z_0)^2]^{3/2}} - \frac{3[x^2 + y^2 + (z - z_0)^2]}{[x^2 + y^2 + (z - z_0)^2]^{5/2}} \right)$$

Now you can see that this really is equal to ZERO

This duplicates the discussion about the Dirac delta function

- At the charges $\mathbf{x} \mp z_0 \hat{\mathbf{k}} = 0$, and $\nabla^2 V(\mathbf{x}) = -\rho(\mathbf{x})/\epsilon_0 \neq 0$

Compare dipole & charge above conducting plane

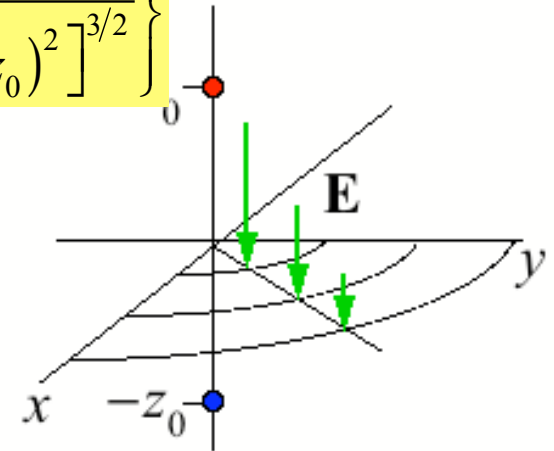
- Dipole field

$$\mathbf{E}(x, y, z) = \frac{q}{4\pi\epsilon_0} \left\{ \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z - z_0)\hat{\mathbf{k}}}{[x^2 + y^2 + (z - z_0)^2]^{3/2}} - \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z + z_0)\hat{\mathbf{k}}}{[x^2 + y^2 + (z + z_0)^2]^{3/2}} \right\}$$

- Dipole field on the midplane is normal to plane

$$\mathbf{E}(x, y, 0) = \frac{-qz_0\hat{\mathbf{k}}}{2\pi\epsilon_0[r^2 + z_0^2]^{3/2}}$$

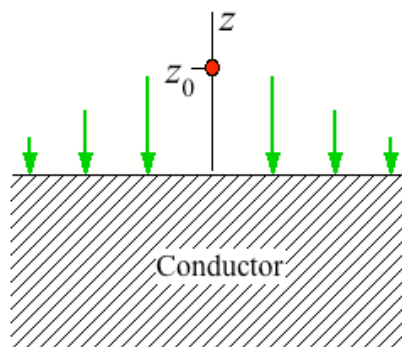
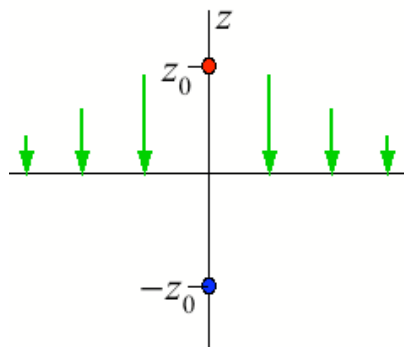
$$r^2 = x^2 + y^2$$



We should have known, $\mathbf{E}$ normal to equipotentials

- 1) Dipole potential satisfies Laplace's equation. ($\mathbf{x} \neq \pm z_0\hat{\mathbf{k}}$)
- 2) Dipole potential is zero on the mid-plane.

Compare with a point charge **above** a grounded conducting plane.



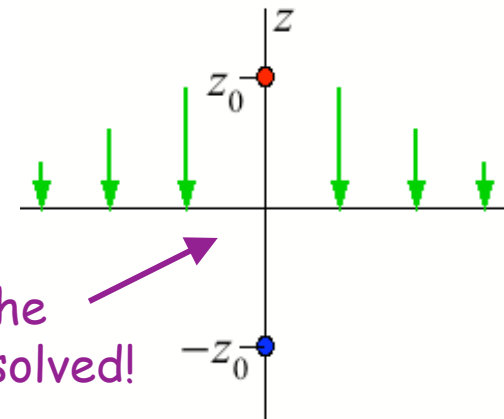
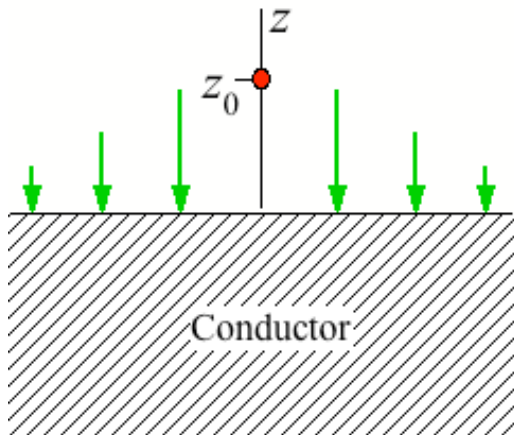
V satisfies Laplace's equation

$V = 0$, on the surface

Suggests a "method of images" for solving Laplace's Eq.

Uniqueness of solutions to Laplace's Eq.

Solving this problem is equivalent to solving this problem



Field of a dipole on the mid plane is already solved!

$\mathbf{E}$ field on conductor's surface = $\mathbf{E}$ field of dipole on the midplane

$$\mathbf{E}(x, y, 0) = \frac{-qz_0 \hat{\mathbf{k}}}{2\pi\epsilon_0 [r^2 + z_0^2]^{3/2}}$$

$$r^2 = x^2 + y^2$$

Conductor's surface charge density

Conductor's total charge

$$\sigma(x, y) = \epsilon_0 E_n(x, y)$$

$$= \frac{-qz_0}{2\pi [r^2 + z_0^2]^{3/2}}$$

$$\oint_S \sigma(x, y) d^2x' = \frac{-qz_0}{2\pi} \int_0^{2\pi} d\phi \int_0^\infty \frac{r dr}{(r^2 + z_0^2)^{3/2}} = -q$$