PHY481: Electromagnetism

Solving Laplace's Equation

Method of Images

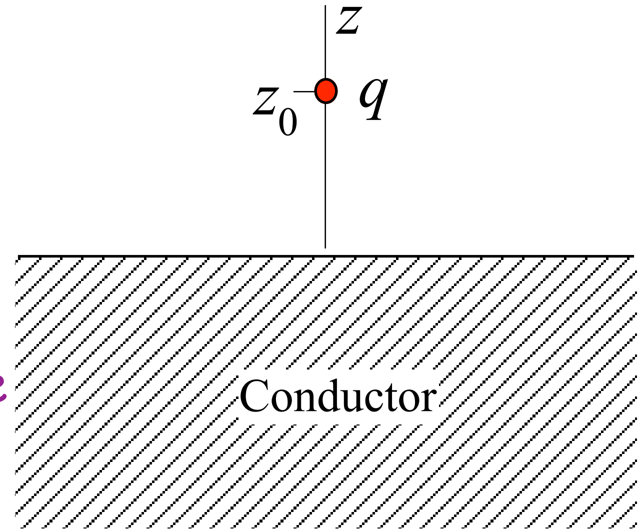
Problems with specific symmetries
Rectangular, Spherical & Cylindrical

What kind of problems need the image method?

V = 0 boundary condition

Charge q above a grounded conductor.

What are the potential & field above, and surface charge density on, the conductor?



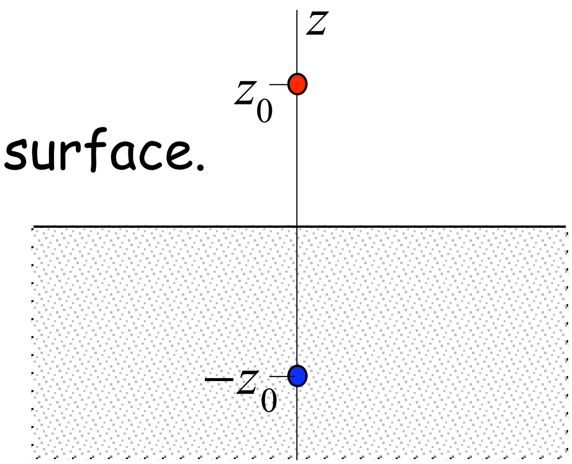
You must recognize that this is a problem where the symmetry implies a solution can be "easily" found via the

Method of Images

Temporarily forget about the conductor

Replace it by reflection of the space above its surface.

If reflected charge is -q, we have a dipole!



Dipole potential is zero on the midplane

just like the V = 0 boundary condition above.

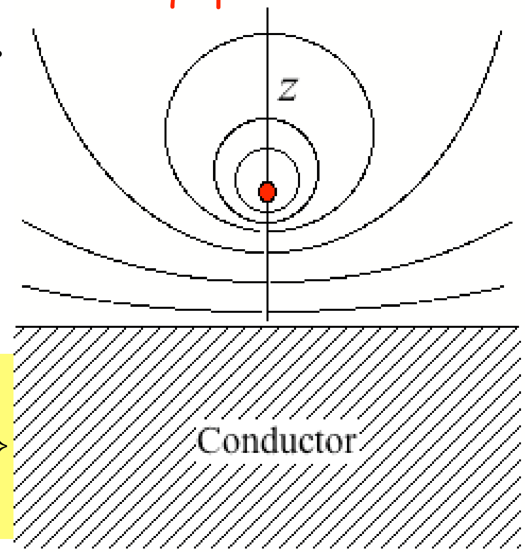
Dipole potential is a solution of Laplace's equation $\left(\mathbf{x} \neq \pm z_0 \hat{\mathbf{k}}\right)$

DONE ?

Almost done

Dipole potential OK, but applies only above the plane.

$$V(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0}\left\{\left[x^2+y^2+(z-z_0)^2\right]^{(-1/2)} - \left[x^2+y^2+(z+z_0)^2\right]^{(-1/2)}\right\}$$

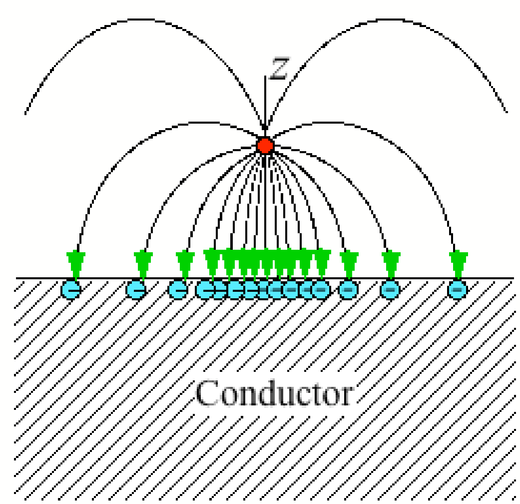


Field above the plane is the dipole field $\mathbf{E} = -\nabla V$

$$\mathbf{E}(x,y,z) = \frac{q}{4\pi\varepsilon_0}\left\{\frac{x\,\hat{\mathbf{i}}+y\,\hat{\mathbf{j}}+(z-z_0)\hat{\mathbf{k}}}{\left[x^2+y^2+(z-z_0)^2\right]^{3/2}} - \frac{x\,\hat{\mathbf{i}}+y\,\hat{\mathbf{j}}+(z+z_0)\hat{\mathbf{k}}}{\left[x^2+y^2+(z+z_0)^2\right]^{3/2}}\right\}$$

Field at the conductor's surface

$$\mathbf{E}(x,y,0) = \frac{-qz_0\,\hat{\mathbf{k}}}{2\pi\varepsilon_0\left[r^2+z_0^2\right]^{3/2}} = E_n\,\hat{\mathbf{k}}$$

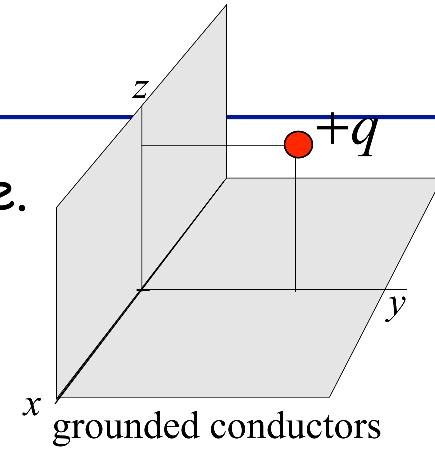


Conductor's surface charge density

$$\sigma(x,y) = \varepsilon_0 E_n(x,y) = \frac{-qz_0}{2\pi\left[r^2+z_0^2\right]^{3/2}}$$

Now you are done!

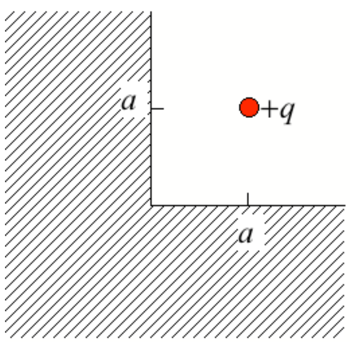
New image problem



Charge *q* above a grounded conductors forming a vee.

What are the potential & field for *y*,*z* > 0 , and surface charge density on the conductors?

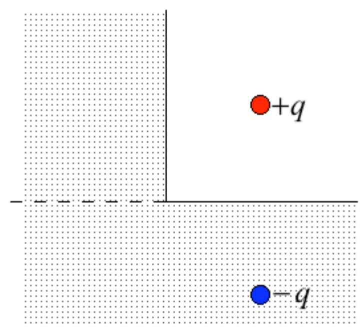
Problem to solve



instead

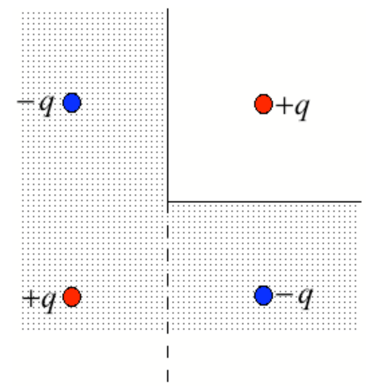
Get potential zero at z = 0

Reflect on horizontal axis



Get potential zero at y = 0

Reflect on vertical axis



Total of +q and image charges is zero. --> +q will draw -q onto conductors

Potential in the notch will be that of opposing dipoles at y = ± a

Potential of opposing dipoles will tend to cancel far from the origin.

This leaves only a "quadrupole" potential in a multipole expansion .

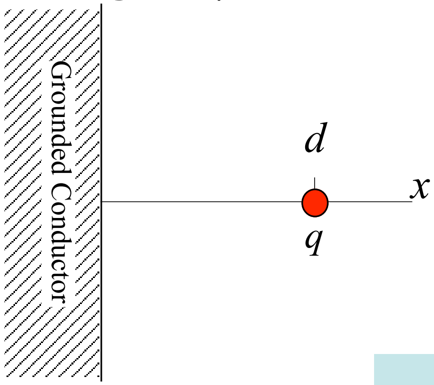
Potential is "saddle shaped" at the corner , -q's at stirrups, +q's in front & back

Practical image problem: time q hits the surface

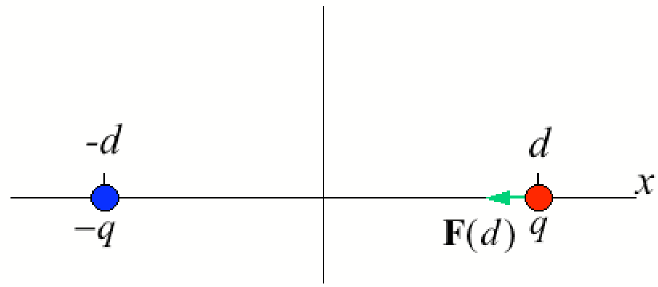
Charge q, mass m is released a distance d from a grounded plane.

Negative charge drawn from ground. Positive charge hits plane at time t.

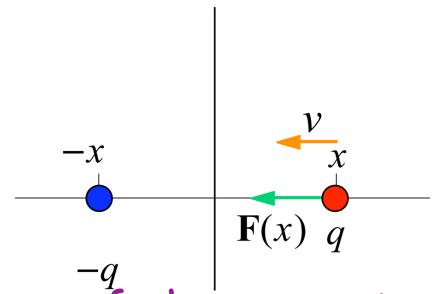
<u>original problem</u>



<u>equivalent image</u>



<u>charges on the move</u>



Get potential energy of charge q at x

Use energy conservation

$$\mathbf{F}(x) = \frac{-q^2}{4\pi\varepsilon_0 (2x)^2}\,\mathbf{i}$$

$$U(x) = -\int_\infty^x \mathbf{F}\cdot\hat{\mathbf{i}}\,dx' = \frac{q^2}{4\pi\varepsilon_0}\int_\infty^x \frac{dx'}{(2x')^2} = \frac{-q^2}{16\pi\varepsilon_0 x}$$

Change in potential energy from d to x

$$\Delta U(x) = U(x) - U(d) = \frac{-q^2}{16\pi\varepsilon_0}\left[\frac{1}{x} - \frac{1}{d}\right]$$

Energy conservation $\Delta KE = -\Delta U(x)$

$$\frac{1}{2}mv^2 = -\Delta U(x) = \frac{q^2}{16\pi\varepsilon_0 d}\left[\frac{d-x}{x}\right]$$

solve for v

$$\frac{dx}{dt} = v = \frac{q}{\sqrt{8\pi\varepsilon_0 md}}\sqrt{\frac{d-x}{x}}$$

integrate over t' and x'

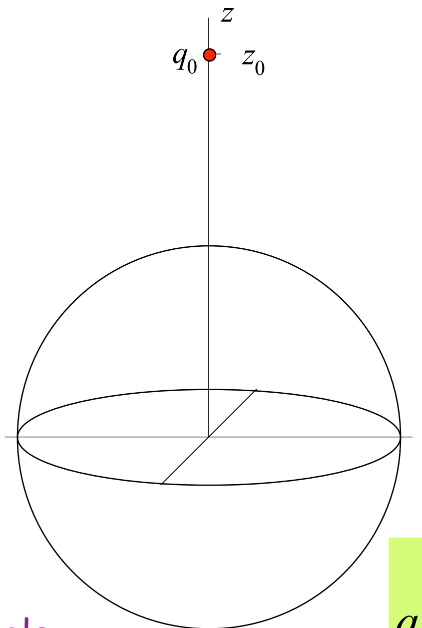
$$\int_0^t dt' = \frac{\sqrt{8\pi\varepsilon_0 md}}{q}\int_d^0 \sqrt{\frac{x'}{d-x'}}\,dx'$$

http://integrals....

$$t = \frac{\sqrt{8\pi\varepsilon_0 md}}{q}\,\frac{\pi d}{2}$$

Spherical image problem

Charge q_0, a distance z_0 from the center of a grounded sphere. Force on q_0?



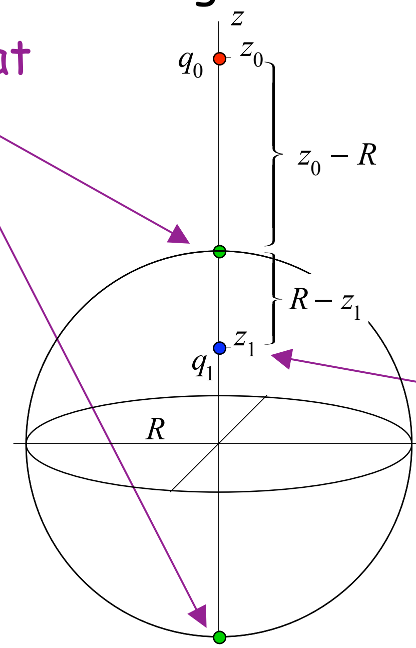
V must be zero at these points

$$\frac{q_0}{z_0 - R} = \frac{-q_1}{R - z_1}$$

$$\frac{q_0}{z_0 + R} = \frac{-q_1}{z_1 + R}$$

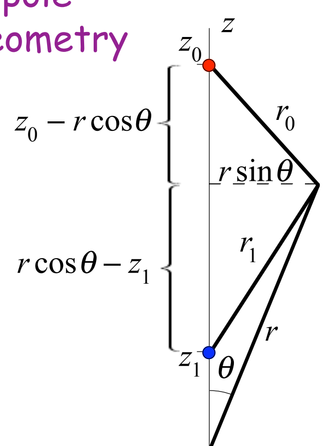
simultaneous eq.

$$q_1 = -q_0 \frac{R}{z_0}; \quad z_1 = \frac{R^2}{z_0}$$



Replace sphere with image charge q_1, a distance z_1 from the center.

Dipole geometry



Two charge potential

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_0}{r_0} + \frac{q_1}{r_1}\right]$$

Force on q_0

$$F_z = \frac{q_0 q_1}{4\pi\varepsilon_0 (z_0 - z_1)^2} = \frac{-q_0^2 (z_0/R)}{4\pi\varepsilon_0 R^2\left[(z_0/R)^2 - 1\right]^2}$$

Field & PE

$$\mathbf{E} = -\nabla V$$

$$W = U = -\int_{\infty}^{z_0} F_z\, dz$$

Solving Laplace's equation in problems with Spherical Symmetry (only r dependence)

Spherical coordinates

$$\nabla^2 V(r) = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV(r)}{dr}\right) = 0$$

Guess a solution

$$V(r) = \frac{A}{r} + B$$

e.g., point charge

$$V = \frac{q}{4\pi\varepsilon_0 r}$$

Don't like to Guess? Determine a solution:

$$\text{Let } \frac{dV(r)}{dr} = f(r)$$

$$r^2\nabla^2 V = \frac{d}{dr}\left(r^2 f(r)\right) = 0$$

$$= 2rf(r) + r^2\frac{df(r)}{dr}$$

$$\frac{df(r)}{f(r)} = -2\frac{dr}{r}$$

$$f(r) = -Ar^{-2}$$

$$\frac{dV(r)}{dr} = -Ar^{-2}$$

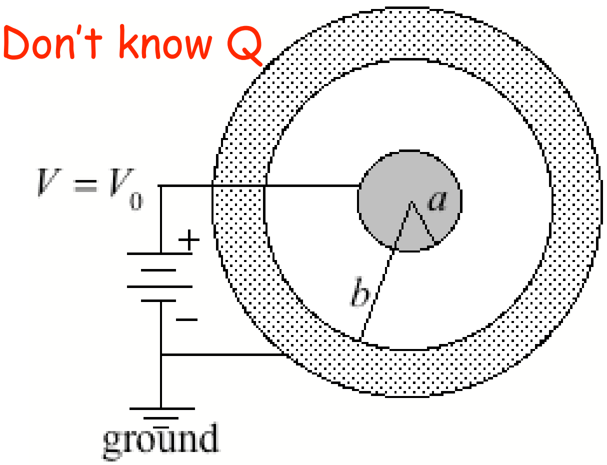
General solution

$$V(r) = Ar^{-1} + B$$

The spherical capacitor

Battery keeps an inner sphere at potential V_0 while outer sphere is grounded. What are potential, field and charge densities?

Don't know Q



General solution

$$V(r) = Ar^{-1} + B$$

Boundary conditions

$$V(a) = V_0; \quad V(b) = 0$$

Apply boundary conditions to get A, B

$$V(a) = \frac{A}{a} + B = V_0 \qquad V(b) = \frac{A}{b} + B = 0$$

$$B = -\left(\frac{a}{b-a}\right)V_0 \qquad A = \left(\frac{ab}{b-a}\right)V_0$$

Complete the potential

$$V(r) = \frac{V_0 a}{r}\left(\frac{b-r}{b-a}\right)$$

Potential clearly satisfies boundary conditions at $r = a$ or b

Calculate the field

$$\mathbf{E} = -\nabla V(r) = \frac{V_0}{r^2}\left(\frac{ab}{b-a}\right)\hat{\mathbf{r}}$$

Charge densities:

$$\sigma_a = \varepsilon_0 E_n(a) = \varepsilon_0 V_0 \frac{b}{a}\frac{1}{b-a}$$

$$\sigma_b = \varepsilon_0 E_n(b) = -\varepsilon_0 V_0 \frac{a}{b}\frac{1}{(b-a)}$$

Charges:

$$Q_a = \left(4\pi a^2\right)\sigma_a = 4\pi\varepsilon_0 V_0 \frac{ab}{b-a} = Q$$

$$Q_b = \left(4\pi b^2\right)\sigma_b = -4\pi\varepsilon_0 V_0 \frac{ab}{b-a} = -Q$$

Solving Laplace's equation in problems with Cylindrical Symmetry (only r dependence)

Cylindrical coordinates

$$\nabla^2 V(r) = \frac{1}{r}\frac{d}{dr}\left(r\frac{dV(r)}{dr}\right) = 0$$

Guess a solution

$$V(r) = A\ln\left(\frac{r}{r_0}\right) + B$$

e.g., Line charge

$$V = \frac{-\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r}{a}\right)$$

Determine a solution

$$\text{Let } \frac{dV(r)}{dr} = f(r)$$

$$r\nabla^2 V = \frac{d}{dr}\left(rf(r)\right) = 0$$

$$= f(r) + r\frac{df(r)}{dr}$$

$$\frac{df(r)}{f(r)} = -\frac{dr}{r}$$

$$f(r) = Ar^{-1} = \frac{dV}{dr}$$

$$\frac{dV(r)}{dr} = Ar^{-1}$$

General solution

$$V(r) = A\ln\left(r/r_0\right) + B$$

Cylindrical capacitor

Battery keeps an inner cylinder at potential V_0 while outer one is grounded. What are potential, field and charge densities?

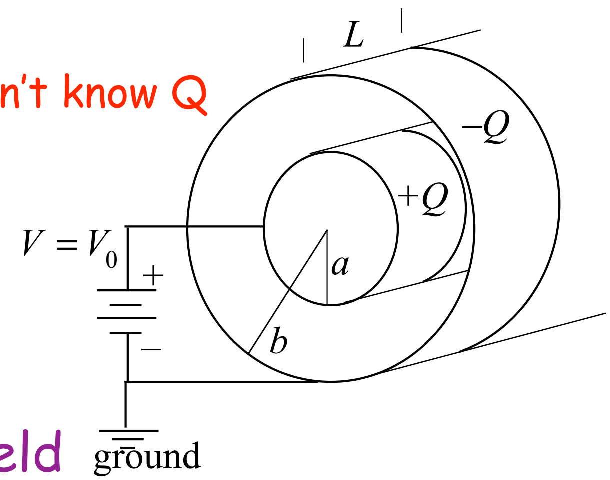
General solution

$$V(r) = A\ln(r/r_0) + B$$

Boundary conditions

$$V(b) = 0; \quad V(a) = V_0$$

Don't know Q



Apply boundary conditions to get A, B

$$0 = A\ln(b/r_0) + B \qquad V_0 = A\ln(a/r_0) + B$$

$$A = -V_0/\ln(b/a) \qquad B = +V_0\ln(b/r_0)/\ln(b/a)$$

Complete potential

$$V(r) = V_0\frac{\ln(b/r)}{\ln(b/a)}$$

boundary conditions clearly satisfied at $r = a$ or b

Calculate field

$$\mathbf{E} = -\nabla V(r) = \frac{V_0}{r\ln(b/a)}\hat{\mathbf{r}}$$

Charge densities

$$\sigma_a = \varepsilon_0 E_n(a) = \frac{\varepsilon_0 V_0}{a\ln(b/a)}$$

$$\sigma_b = \varepsilon_0 E_n(b) = -\frac{\varepsilon_0 V_0}{b\ln(b/a)}$$

Charge/unit L

$$\tilde{Q}_a = (2\pi a)\sigma_a = \frac{2\pi\varepsilon_0 V_0}{\ln(b/a)}$$

$$\tilde{Q}_b = (2\pi b)\sigma_b = -\frac{2\pi\varepsilon_0 V_0}{\ln(b/a)}$$

Capacitance/unit L

$$\tilde{C} = \frac{2\pi\varepsilon_0}{\ln(b/a)}$$

Solving Laplace's equation in problems with Angular Dependence (e.g., point dipole)

Spherical coordinates (polar angle θ)

$$\nabla^2 V(r,\theta) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) + \frac{1}{r\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right) = 0$$

Guess a solution (one of many)

$$V = Ar^{-1} + B + \frac{C\cos\theta}{r^2} + Dr\cos\theta$$

e.g., point dipole

$$V = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

or

linear V

$$V = E_0 z$$

Cylindrical coordinates (azimuth angle ϕ)

$$\nabla^2 V(r,\phi) = \frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) + \frac{1}{r^2}\frac{d^2V}{d\phi^2} = 0$$

Guess a solution (one of many)

$$V = A\ln\left(\frac{r}{r_0}\right) + B + \frac{C\cos\phi}{r} + Dr\cos\phi$$

e.g., line dipole

$$V = \frac{E_0 a^2\cos\theta}{r}$$

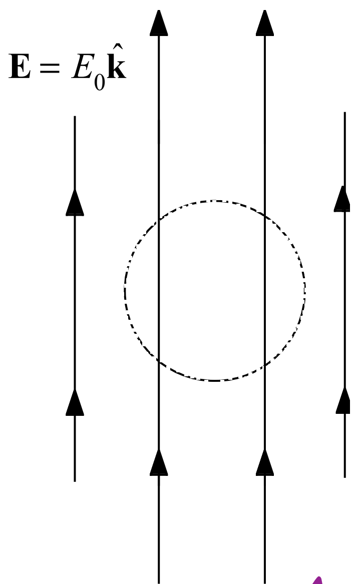
or

linear V

$$V = E_0 x$$

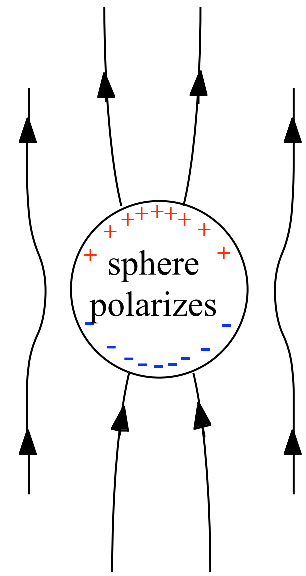
Classic problem

Before



Grounded conducting sphere is inserted into a uniform field in z direction. What is the new field and charge density?

After



General solution

$$V(r,\theta) = \frac{A}{r} + B + \frac{C\cos\theta}{r^2} + Dr\cos\theta$$

Boundary conditions

$$Q = 0 \quad \Rightarrow \quad A = 0$$

$$V(a,\theta) = 0; \quad V(r,\theta)\big|_{r\to\infty} = -E_0 z$$

Apply boundary conditions

$$B = 0 \quad \text{arbitrary}$$

$$V(r,\theta)\big|_{r\to\infty} = -E_0 z$$

$$D = -E_0 \ (z = r\cos\theta)$$

$$V(a,\theta) = 0 = \frac{C\cos\theta}{a^2} - E_0 a\cos\theta$$

$$C = E_0 a^3$$

Complete solution

$$V(r,\theta) = E_0 r\cos\theta\left(\frac{a^3}{r^3} - 1\right)$$

Field is radial at r = a.

$$\mathbf{E} = E_0\left(1 + \frac{2a^3}{r^3}\right)\cos\theta\,\hat{\mathbf{r}} + E_0\left(1 - \frac{a^3}{r^3}\right)\sin\theta\,\hat{\boldsymbol{\theta}}$$

Charge density

$$\sigma(\theta) = 3\varepsilon_0 E_0\cos\theta$$