PHY481: Electromagnetism

Solving Laplace's Equation

Problems with specific symmetries (summary)

Solutions via "Separation of Variables"

Cartesian coordinates: Fourier Series

Summary: Solving Laplace's equation in problems with specific symmetries

1) Grounded conductor and an external charge or dipole:

Potential solved by the Method of Images

Rectangular, Spherical, and Cylindrical symmetry

2) Potential between conductors: capacitors $Q = CV$

Parallel plate capacitors $V(z) = Az + B$

Spherical capacitors $V(r) = Ar^{-1} + B$

Cylindrical capacitors $V(r) = A\ln(r/r_0) + B$

3) Potentials for conductors in external fields

Spherical symmetry $$V(r,\theta) = Ar^{-1} + B + \frac{C\cos\theta}{r^2} + Dr\cos\theta$$

Cylindrical symmetry $$V(r,\phi) = A\ln\left(\frac{r}{r_0}\right) + B + \frac{C\cos\phi}{r} + Dr\cos\phi$$

Solving Laplace's equation via "Separation of Variables"

In general, a solution of Laplace's equation subject to boundary conditions requires an infinite series of functions.

Cartesian coordinate boundary conditions: Fourier series

Any periodic function with period 2a, can be expanded in a "Fourier Series" of sine and cosine functions, with n a positive integer

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{a} + b_n \sin\frac{n\pi x}{a} \right]$$

"Orthogonality" of the trigonometric functions

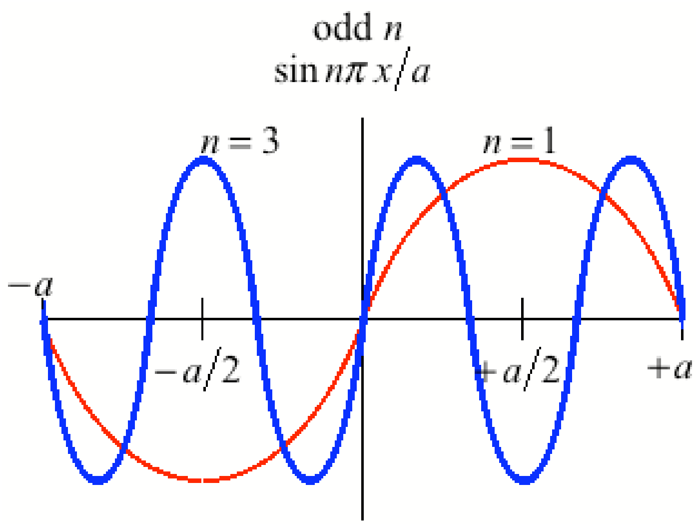
$$\frac{1}{a}\int_{-a}^{a} \cos\frac{n\pi x}{a}\cos\frac{m\pi x}{a} = \delta_{nm} \qquad \frac{1}{a}\int_{-a}^{a} \sin\frac{n\pi x}{a}\sin\frac{m\pi x}{a} = \delta_{nm}$$

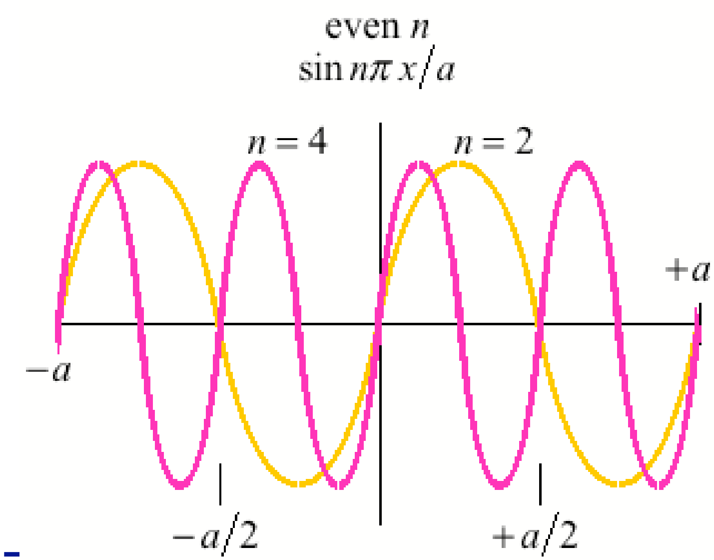
The coefficients a_n and b_n are determined by doing these integrals

$$a_0 = \frac{1}{2a}\int_{-a}^{a} f(x)dx \qquad a_m = \frac{1}{a}\int_{-a}^{a} f(x)\cos\frac{m\pi x}{a}dx \qquad b_m = \frac{1}{a}\int_{-a}^{a} f(x)\sin\frac{m\pi x}{a}dx$$

Odd or even functions

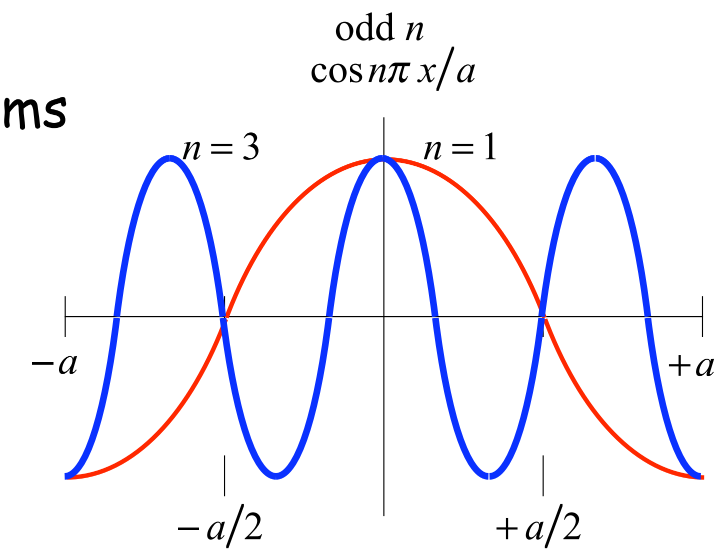
Odd functions select these



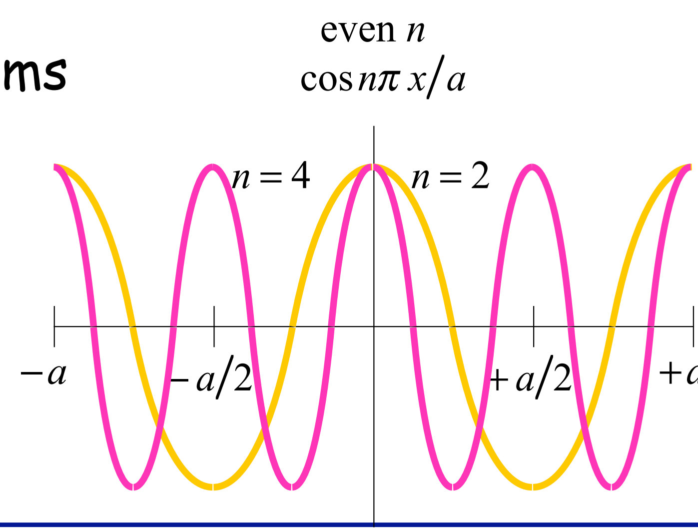


Even functions select these

Odd n terms

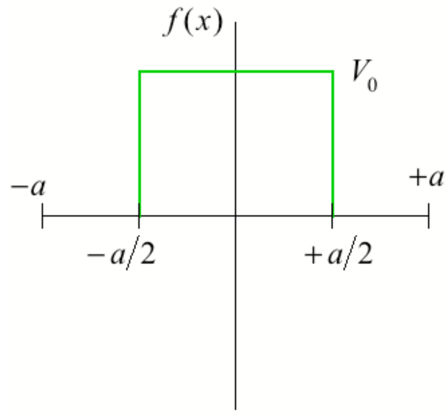


Even n terms

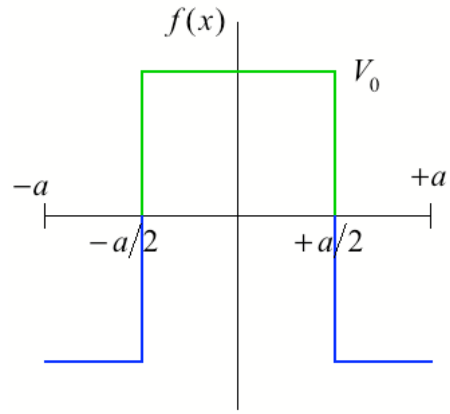


Qualitative expansion of $f(x) = V_0$

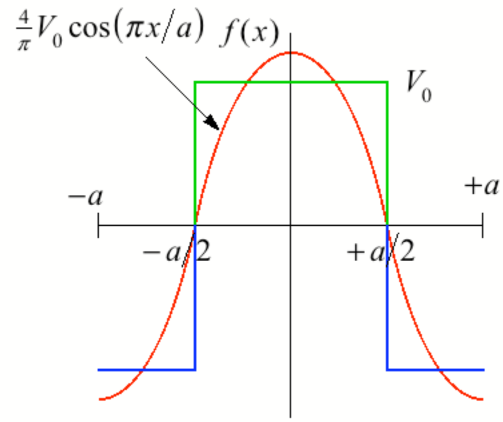
Fourier series expansion
$f(x) = V_0$ $(-a/2 < x < +a/2)$



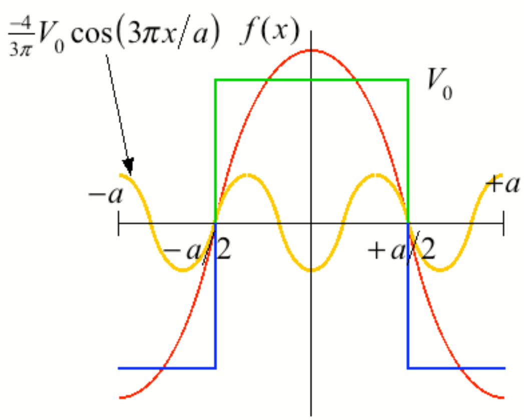
Extend f(x) to make
a periodic function



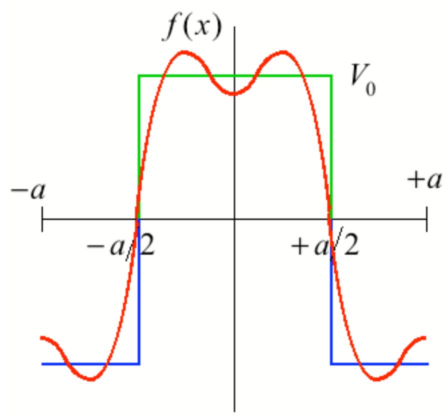
Note similarity to



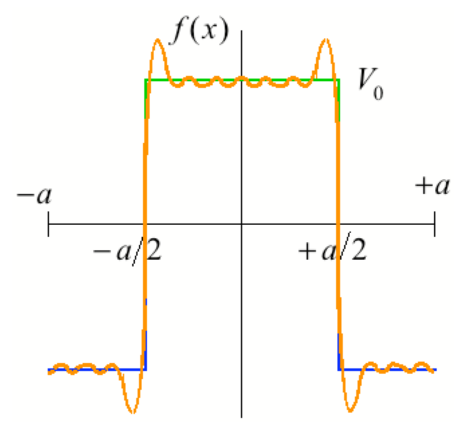
Prepare a fraction
of the next odd n cosine



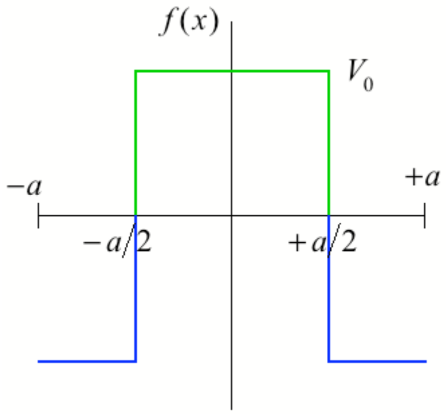
Add to the first.
Getting closer!



Add a few more terms
Pretty good match!



Coefficients in a Fourier Series



Find Fourier series expansion of $f(x) = V_0$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{a} + b_n \sin\frac{n\pi x}{a}\right]$$

$$a_m = \frac{1}{a}\int_{-a}^{a} f(x)\cos\frac{m\pi x}{a}$$

$$b_m = \frac{1}{a}\int_{-a}^{a} f(x)\sin\frac{m\pi x}{a} = 0$$

function f(x) is even
so no sine terms

$$= \frac{1}{m\pi}\left[-V_0 \sin\frac{m\pi x}{a}\bigg|_{-a}^{-a/2} + V_0 \sin\frac{m\pi x}{a}\bigg|_{-a/2}^{+a/2} - V_0 \sin\frac{m\pi x}{a}\bigg|_{+a/2}^{+a}\right]$$

$$a_m = \begin{cases} \dfrac{4V_0}{m\pi}(-1)^n, m = 2n+1 \text{ (odd)} \\ 0, m = 2,4,\ldots \end{cases}$$

$$f(x) = \frac{4V_0}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\cos\left[(2n+1)\frac{\pi x}{a}\right]$$

Solution of ∇²V=0 by "Separation of Variables"

Laplace's equation for an electric potential in Cartesian coordinates

$$\nabla^2 V(\mathbf{x}) = \frac{\partial^2 V(\mathbf{x})}{\partial x^2} + \frac{\partial^2 V(\mathbf{x})}{\partial y^2} + \frac{\partial^2 V(\mathbf{x})}{\partial z^2} = 0$$

Best with rectangular boundary conditions

Solutions are often of the "separable" type,

$$V(\mathbf{x}) = X(x)Y(y)Z(z)$$

or linear combinations of such solutions:

$$V(\mathbf{x}) = \sum_{n=0}^{\infty} V_n(\mathbf{x});$$

$$V_n(\mathbf{x}) = X_n(x)Y_n(y)Z_n(z))$$

Partial derivatives in Laplace's equation become total derivatives

$$\nabla^2 V(\mathbf{x}) = \frac{d^2 X(x)}{dx^2} YZ + \frac{d^2 Y(y)}{dy^2} XZ + \frac{d^2 Z(z)}{dz^2} XY = 0$$

Divide by V

$$\nabla^2 V(\mathbf{x}) / V(\mathbf{x}) = \frac{1}{X}\frac{d^2 X(x)}{dx^2} + \frac{1}{Y}\frac{d^2 Y(y)}{dy^2} + \frac{1}{Z}\frac{d^2 Z(z)}{dz^2} = 0$$

Coordinate independence in separable solutions

Cartesian coordinate separable solutions $V(\mathbf{x}) = X(x)Y(y)Z(z)$

Translational independence x --> x' $V(\mathbf{x}') = X(x')Y(y)Z(z)$

$$\nabla^2 V / V = \frac{1}{X(x')}\frac{d^2 X(x')}{dx^2} + \frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2 Z(z)}{dz^2} = 0$$

Values of the last two terms (Y and Z terms) have not changed.

Yet the sum is still ZERO. Each of the 3 terms must be a constant.

$$\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} = k_1^2 \qquad \frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2} = k_2^2 \qquad \frac{1}{Z(z)}\frac{d^2 Z(z)}{dz^2} = k_3^2$$

Solution to Laplace's equation must have $k_1^2 + k_2^2 + k_3^2 = 0$

Range of solutions

$$k_1^2 + k_2^2 + k_3^2 = 0$$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = k_1^2; \quad \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = k_2^2; \quad \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = k_3^2$$

If $k_1 = 0$ (simple field) function X satisfies

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = 0$$

solution: $X(x) = ax + b$

Constants a & b determined by the boundary conditions

If $k_1^2 > 0$, function $X(x)$ satisfies

$$\frac{d^2X(x)}{dx^2} = k_1^2 X(x)$$

solution: $X(x) = Ae^{k_1 x} + Be^{-k_1 x}$

etc., for Y & Z

If $k_1^2 < 0$, then $k \to ik'$ k' real > 0, and X satisfies

$$\frac{d^2X(x)}{dx^2} = -k_1'^2 X(x)$$

solution: $X(x) = Ae^{ik_1' x} + Be^{-ik_1' x}$

etc., for Y & Z

Character of solutions

$k_1^2 > 0$ $X(x) = Ae^{k_1 x} + Be^{-k_1 x}$

Allowable values of A, B, and k_1 determined by symmetries and boundary conditions

If region unbounded +x $A = 0, \quad X(x) = Be^{-k_1 x}$

If region unbounded -x $B = 0, \quad X(x) = Ae^{k_1 x}$

If region bounded both + & -

$X(x) = A'\cosh k_1 x + B'\sinh k_1 x$

Even symmetry Odd symmetry

$$\cosh k_1 x = \left(\frac{e^{k_1 x} + e^{-k_1 x}}{2}\right); \quad \sinh k_1 x = \left(\frac{e^{k_1 x} - e^{-k_1 x}}{2}\right)$$

$k_1^2 < 0$

$X(x) = Ae^{ik_1' x} + Be^{-ik_1' x}$

$X(x) = A'\cos k_1' x + B'\sin k_1' x$

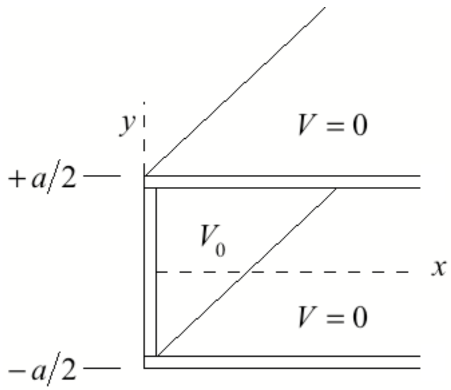
Suggests using Fourier series for A, B, k_1

Even symmetry Odd symmetry

$$\cos k_1 x = \left(\frac{e^{ik_1 x} + e^{-ik_1 x}}{2}\right); \quad \sin k_1 x = \left(\frac{e^{ik_1 x} - e^{-ik_1 x}}{2i}\right)$$

Example: Long narrow channel

Large top and bottom grounded plates, potential V_0 on the end plate
Find potential everywhere between the plates.



$$V(x,y) = X(x)Y(y)$$

problems without *z* dependence all start the same way

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = 0$$

Laplace's equation

Each term must be a constant

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = k_1^2 \qquad \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = k_2^2$$

$$k_1^2 + k_2^2 = 0$$

$$k_1^2 = +k^2, \quad k_2^2 = -k^2$$

Even y symmetry

<u>First determine Y</u>

$$\frac{d^2Y(y)}{dy^2} = -k^2Y(y)$$

$$Y(y) = A\cos ky + B\sin ky$$

$$Y(y) = A\cos ky \quad B = 0$$

V = 0 on top and bottom plate

$$Y(a/2) = \cos(ka/2) = 0$$

$$k = (2n+1)\frac{\pi}{a}; \quad n = 0, 1, 2, \ldots$$

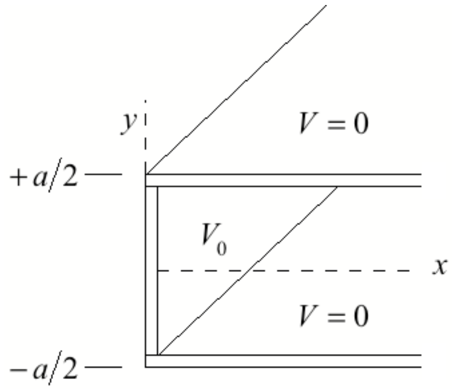
Any one of these values for *k*, satisfies the boundary condition at top and bottom, But an infinite series is needed to satisfy $V(0,y) = V_0$

Solution with coefficients

$$Y(y) = \sum_{n=0}^{\infty} A_n \cos\left[(2n+1)\frac{\pi y}{a}\right]$$

Example (cont'd)

Determine the X function



$$\frac{d^2X(x)}{dx^2} = +k^2X(x)$$

General solution $X(x) = Ae^{kx} + Be^{-kx}$

Finite at large +x $X(x\to\infty)=0, \quad A=0$ $\qquad X(x) = Be^{-kx}$

From Y solution $k = (2n+1)\dfrac{\pi}{a}; \quad n = 0,\ 1,\ 2,\ \ldots$ $\qquad X(x) = Be^{-(2n+1)\pi x/a}$

V = XY and determine the coefficients $\qquad C_n = A_nB$

$$V(x,y) = X(x)Y(y) = \sum_{n=0}^{\infty} C_n \cos\frac{(2n+1)\pi y}{a} e^{-(2n+1)\pi x/a}$$

On left boundary, Potential is V_0

$$V(0,y) = V_0 = \sum_{n=0}^{\infty} C_n \cos\left[\frac{(2n+1)\pi y}{a}\right]$$

Fourier tells us we get C_n this way

$$C_n = \frac{1}{a}\int_{-a}^{a} V_0 \cos\frac{(2n+1)\pi y}{a} = \frac{4V_0(-1)^n}{(2n+1)\pi}$$

Finally a solution

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\cos\frac{(2n+1)\pi y}{a} e^{-(2n+1)\pi x/a}$$