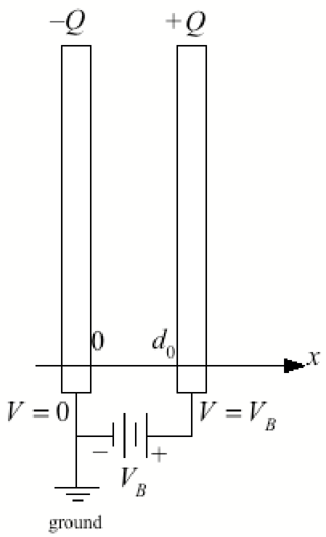
PHY481: Electromagnetism

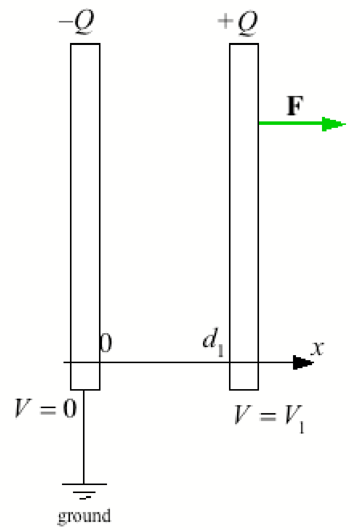
HW4

4.3

Plates charged

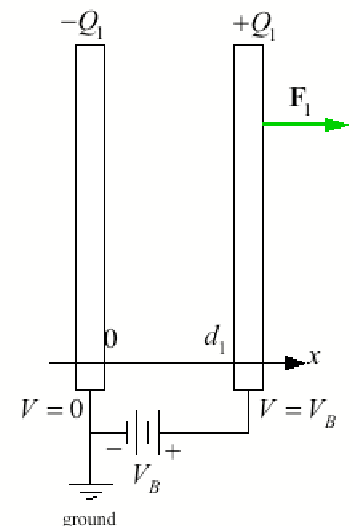


No battery — Spread plates — With battery



Charge remains at Q

Potential changes to V_1



Charge changes to Q_1

Potential remains at V_B

$$Q = CV_B = \varepsilon_0 A V_B / d_0$$

$$Q = C_1 V_1 = \varepsilon_0 A V_1 / d_1$$

$$V_1 = V_B\, d_1 / d_0$$

$$Q_1 = C_1 V_B = \varepsilon_0 A V_B / d_1$$

$$Q_1 = Q d_0 / d_1$$

Stored energy

$$\mathcal{E} = \frac{1}{2} C V_B^2 = \frac{1}{2} \frac{\varepsilon_0 A}{d_0} V_B^2$$

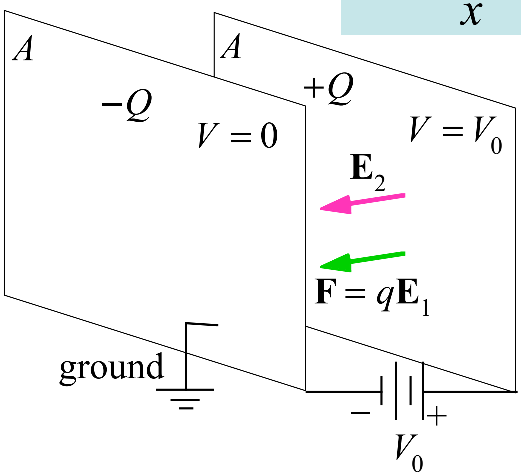
$$\mathcal{E}_1 = \frac{1}{2} C_1 V_1^2 = \frac{d_1}{d_0} \mathcal{E}$$

$$\mathcal{E}_1' = \frac{1}{2} C_1 V_B^2 = \frac{d_0}{d_1} \mathcal{E}$$

?

4.3

$Q = CV$ $\quad$ $C = \dfrac{\varepsilon_0 A}{x}$



Field between the plates is the sum of the field of both plates $\quad$ $\mathbf{E}_2 = \sigma/\varepsilon_0$

Force on +Q generated by field of the -Q charge on one plate $\quad$ $\mathbf{E}_1 = \sigma/2\varepsilon_0$

	Battery disconnected	Battery connected
	$Q = Q_0$	$V = V_0$
$\mathbf{F} = q\mathbf{E}_1 = Q\dfrac{\sigma}{2\varepsilon_0} =$	$\dfrac{Q_0^2}{2\varepsilon_0 A} = \dfrac{\varepsilon_0 A V_0^2}{2x_0^2}$	$\dfrac{\varepsilon_0 A V_0^2}{2x^2}$
Work done to pull plates apart by Δx: $W = -\displaystyle\int_{x_0}^{x_0+\Delta x} \mathbf{F}\cdot\hat{\mathbf{i}}\,dx =$	$\dfrac{\varepsilon_0 A V_0^2}{2}\dfrac{\Delta x}{x_0^2}$	$\dfrac{\varepsilon_0 A V_0^2}{2}\dfrac{\Delta x}{x_0(x_0+\Delta x)}$
Cap. energy storage $U = \dfrac{Q^2}{2C}$ Energy change $\Delta U = \dfrac{Q^2}{2C} - \dfrac{Q_0^2}{2C_0} =$	$\dfrac{\varepsilon_0 A V_0^2}{2}\dfrac{\Delta x}{x_0^2}$	negative $-\dfrac{\varepsilon_0 A V_0^2}{2}\dfrac{\Delta x}{x_0(x_0+\Delta x)}$
Energy put into the Battery (Δε), where W = ΔU + Δε . $\Delta\mathcal{E} = \Delta Q V_0 =$	0	$\varepsilon_0 A V_0^2\dfrac{\Delta x}{x_0(x_0+\Delta x)}$

4.4

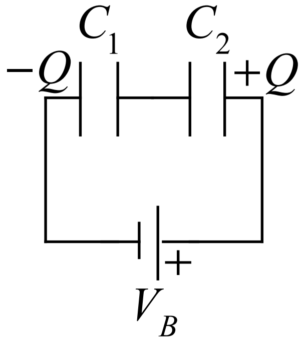
Capacitors in Series

$$V_B = V_1 + V_2; \quad Q = Q_1 = Q_2$$

$$\frac{Q}{C} = \frac{Q}{C_1} + \frac{Q}{C_2}$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$\frac{1}{C} = \sum_{i=1}^{n} \frac{1}{C_i}$$



Capacitors in Parallel

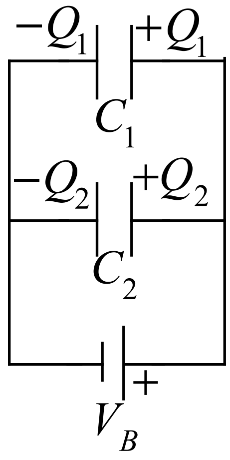
$$Q = Q_1 + Q_2; \quad V_B = V_1 = V_2;$$

$$= C_1 V_B + C_2 V_B$$

$$CV_B = \left(C_1 + C_2\right) V_B$$

$$C = C_1 + C_2$$

$$C = \sum_{i=1}^{n} C_i$$

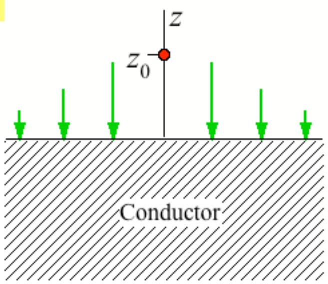


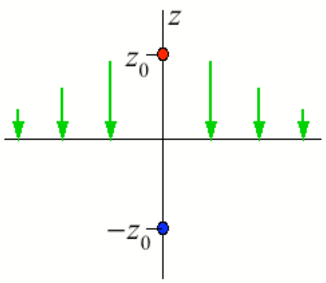
4.6

Potential of charge over conducting plane, via MoI is a discrete dipole potential

$$V(\rho,z)=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{\left(\rho^2+(z-z_0)^2\right)}}-\frac{1}{\sqrt{\left(\rho^2+(z+z_0)^2\right)}}\right]$$

Radius on the mid-plane $\quad \rho^2=x^2+y^2$





Electric field on the mid-plane

$$E_z(\rho,0)=-\left.\frac{\partial V}{\partial z}\right|_{z=0}=-\frac{q}{2\pi\varepsilon_0}\frac{z_0}{\left(\rho^2+z_0^2\right)^{3/2}}$$

Charge density on the conductor surface

$$\sigma(\rho)=\varepsilon_0 E_n(\rho,0)=-\frac{q}{2\pi}\frac{z_0}{\left(\rho^2+z_0^2\right)^{3/2}}$$

Total charge on the on conductor surface

$$Q=2\pi\int_0^\infty\sigma(\rho)\rho d\rho=-qz_0\int_0^\infty\frac{\rho d\rho}{\left(\rho^2+z_0^2\right)^{3/2}}$$

$$=-qz_0\left[\frac{-1}{\left(\rho^2+z_0^2\right)^{1/2}}\right]_0^\infty=-q \quad \left(\text{as expected}\right)$$

Radius enclosing half the surface charge

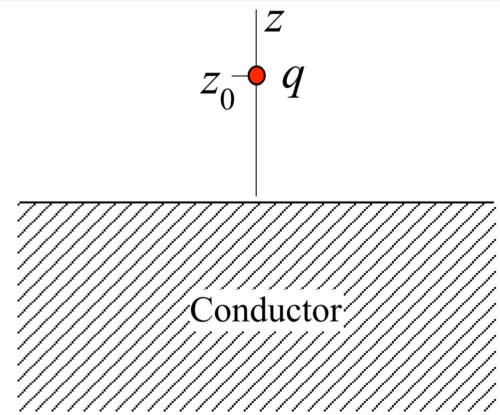
$$\frac{-q}{2}=2\pi\int_0^r\sigma(\rho)\rho d\rho=qz_0\left[\frac{1}{\left(\rho^2+z_0^2\right)^{1/2}}\right]_0^r$$

$$\frac{1}{2}=\frac{1}{\left(x^2+1\right)^{1/2}};\quad x=r/z_0$$

$$x=r/z_0;\quad x=\sqrt{3};\quad r=\sqrt{3}z_0$$

4.7

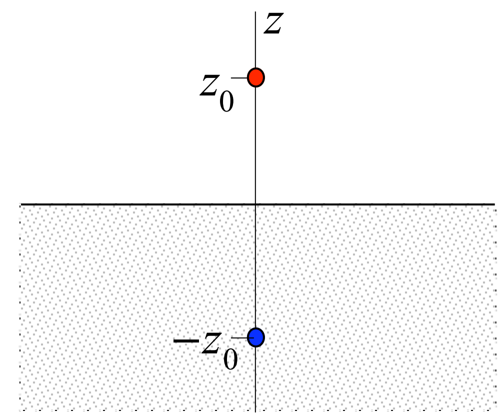
Force on $+q$ from negative charge drawn onto the conductor



Force on $+q$ from negative charge $-q$ in the image.

$$F = \frac{q^2}{4\pi\varepsilon_0 r^2} = \frac{q^2}{4\pi\varepsilon_0 (2z_0)^2} = \frac{q^2}{16\pi\varepsilon_0 z_0^2}$$

$$\mathbf{F} = -\frac{q^2}{16\pi\varepsilon_0 z_0^2}\hat{\mathbf{k}}$$



Energy in field is work done to move $+q$ to the point z_0

$$U = W = -\int_\infty^{z_0} \mathbf{F}\cdot\hat{\mathbf{k}}dr = \frac{q^2}{16\pi\varepsilon_0}\int_\infty^{z_0}\frac{dr}{r^2} = \frac{q^2}{16\pi\varepsilon_0}\left[-\frac{1}{r}\right]_\infty^{z_0}$$

$$= \frac{-q^2}{16\pi\varepsilon_0 z_0}$$

Another way to get field energy. Use delta function charge distribution

$$U = \frac{1}{2}\int \rho(\mathbf{x})V(\mathbf{x})d^3x$$

$$= \frac{1}{2}\int q\delta^3(\mathbf{x} - z_0\hat{\mathbf{k}})\frac{-q}{4\pi\varepsilon_0|\mathbf{x} + z_0\hat{\mathbf{k}}|}d^3x = \frac{-q^2}{8\pi\varepsilon_0|z_0\hat{\mathbf{k}} + z_0\hat{\mathbf{k}}|}$$

$$= \frac{-q^2}{16\pi\varepsilon_0 z_0}$$

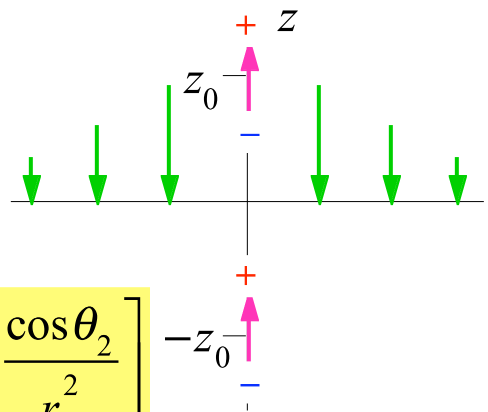
4.10a

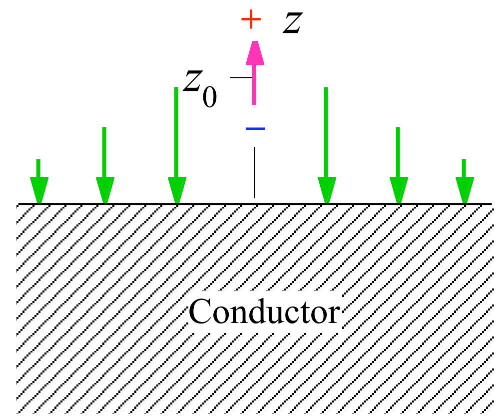
<u>Find potential of an upward facing dipole above a grounded conductor</u>

Potential via M.o.I is two dipoles (SAME DIRECTION)

Single dipole potential

$$V_d(r,\theta) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$





Double dipole potential

$$V_1(r,\theta) + V_2(r,\theta) = \frac{p}{4\pi\varepsilon_0}\left[\frac{\cos\theta_1}{r_1^2} + \frac{\cos\theta_2}{r_2^2}\right]$$

Double dipole potential on z-axis for $z \gg z_0$

$$V_{2d}(z,0) = \frac{p}{4\pi\varepsilon_0}\left[\frac{1}{(z-z_0)^2} + \frac{1}{(z+z_0)^2}\right] \qquad \theta_1 = \theta_2 = 0$$

$$V_{2d}(z,0) = \frac{p}{2\pi\varepsilon_0 z^2}, \quad z >> z_0$$

$$\mathbf{E} = -\Delta V = \frac{p}{\pi\varepsilon_0 z^3}\hat{\mathbf{k}}$$

4.14

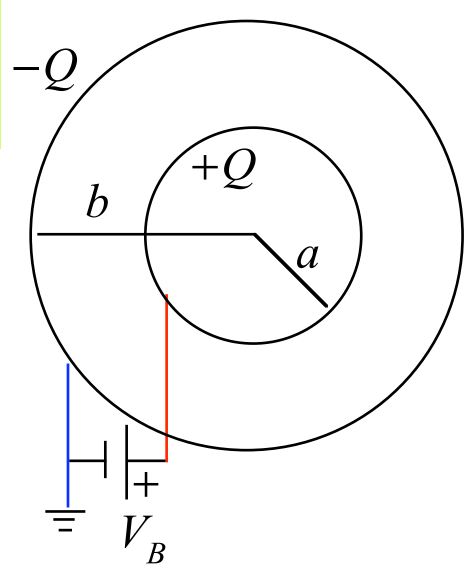
Use Gauss's Law to determine the field and potential between shells.

$$\int_S \mathbf{E}\cdot d\mathbf{A} = \frac{q_{enc}}{\varepsilon_0} \rightarrow \quad \mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$

$$V(r) = -\int_b^r \mathbf{E}\cdot\hat{\mathbf{r}}\, dr = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r}-\frac{1}{b}\right]$$

Determine the field energy

$$U = \frac{\varepsilon_0}{2}\int E^2 d^3x = \frac{Q^2}{8\pi\varepsilon_0}\int_a^b \frac{1}{r^4} r^2 dr = \frac{Q^2}{8\pi\varepsilon_0}\left(-\frac{1}{r}\right)_a^b = \boxed{\frac{Q^2}{8\pi\varepsilon_0}\left(\frac{b-a}{ab}\right)}$$



Determine the capacitance

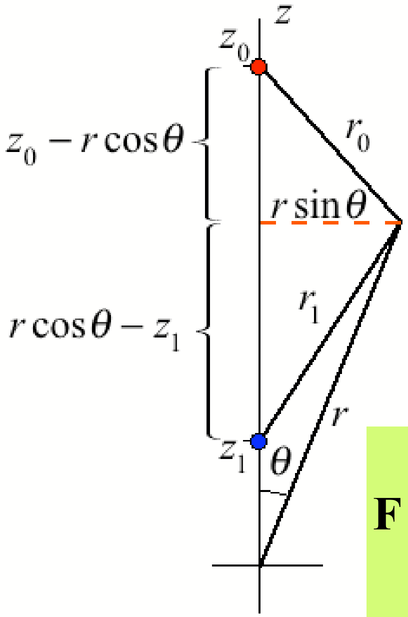
$$U = \frac{1}{2}CV_B^2; \quad V_B = V(a) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{a}-\frac{1}{b}\right]$$

$$C = \frac{2U}{V_B^2} = \frac{\frac{Q^2}{4\pi\varepsilon_0}\left(\frac{b-a}{ab}\right)}{\left[\frac{Q}{4\pi\varepsilon_0}\left(\frac{b-a}{ab}\right)\right]^2} = \boxed{4\pi\varepsilon_0\frac{ab}{b-a}}$$

4.15

<ins>Potential via M.o.I uses</ins> q_2 at z_2 Determine them using points a,b

Geometry



$$V_a = 0 \qquad V_b = 0$$

$$\frac{q_0}{z_0 - R} = \frac{-q_1}{R - z_1} \qquad \frac{q_0}{z_0 + R} = \frac{-q_1}{z_1 + R}$$

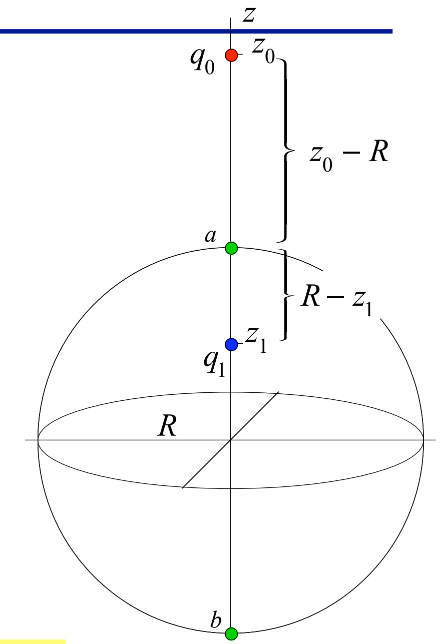
Cross multiply

$$q_0\left(R - z_1\right) = -q_1\left(z_0 - R\right)$$

$$q_0\left(z_1 + R\right) = -q_1\left(z_0 + R\right)$$

Image

$$q_1 = -q_0\frac{R}{z_0}; \quad z_1 = \frac{R^2}{z_0}$$



$$\mathbf{F} = \frac{-q_0^2 R z_0}{4\pi\varepsilon_0\left(z_0^2 - R^2\right)^2}\hat{\mathbf{k}}$$

$$U = -\int_\infty^{z_0} F_z\,dz = -\frac{q_0^2 R}{4\pi\varepsilon_0}\int_\infty^{z_0}\frac{z\,dz}{\left(z^2 - R^2\right)^2} = -\frac{q_0^2 R}{8\pi\varepsilon_0}\frac{1}{\left(z_0^2 - R^2\right)}$$

$$r_0 = \sqrt{r^2 + z_0^2 - 2r\,z_0\cos\theta}$$

$$r_1 = \sqrt{z_0^2 r^2 + R^4 - 2r\,z_0 R^2\cos\theta}\Big/z_0$$

$$E_r(r,\theta) = -\frac{\partial V}{\partial r} = \frac{q}{4\pi\varepsilon_0}\left[\frac{r - z_0\cos\theta}{r_0^3} + \frac{R\left(z_0^2 r - z_0 R^2\cos\theta\right)}{r_1^3}\right]$$

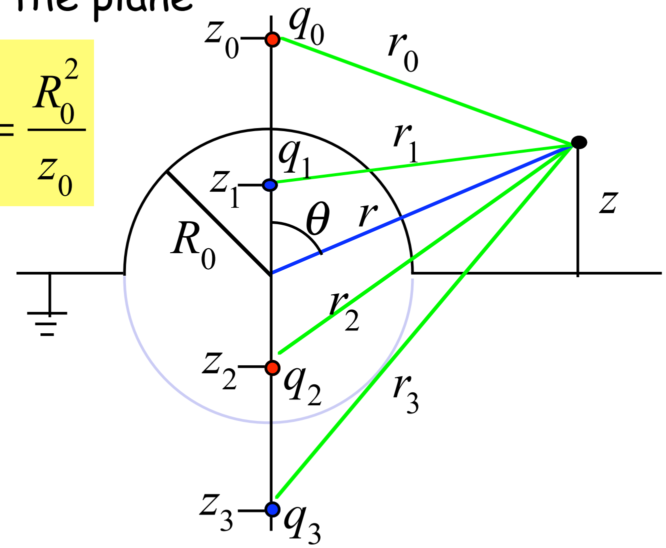
$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_0}{r_0} + \frac{q_1}{r_1}\right]$$

$$E_\theta(r,\theta) = -\frac{\partial V}{r\partial\theta} = \frac{q z_0\sin\theta}{4\pi\varepsilon_0}\left[\frac{1}{r_0^3} + \frac{R^3/z_0^3}{r_1^3}\right]$$

4.16

Charge q_0 above a hemispherical "blister" on a flat grounded plane.

Use Method of Images to determine potential above the plane



1) Get V = 0 on blister surface.

See problem 4.15 for image of q_0 : $q_1 = -q_0 \dfrac{R_0}{z_0}; \quad z_1 = \dfrac{R_0^2}{z_0}$

2) Get V = 0 on plane.

q_2 is image of q_1 : $q_2 = q_0 \dfrac{R_0}{z_0}; \quad z_2 = -\dfrac{R_0^2}{z_0}$

q_3 is image of q_0 : $q_3 = -q_0; \quad z_3 = -z_0$

2) V(r,θ) is sum of the point potentials.

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\sum_{i=0}^{3}\left[\frac{q_i}{r_i}\right] \qquad r_i = |\mathbf{x} - \mathbf{x}_i| = \left[r^2 + z_i^2 - 2rz_i\cos\theta\right]^{1/2}$$

3) Answer given suggests that the easiest way to find the hemisphere's charge is to subtract the charge on the plane from the total charge: $q_1 + q_2 + q_3 = -q_0$

On the plane the normal component of the field is in the direction $\hat{\boldsymbol{\theta}}\,\big|_{\theta=\frac{\pi}{2}} = -\hat{\mathbf{k}}$

$$E_z(r,\pi/2) = -E_\theta(r,\pi/2)$$

$$E_\theta = -\frac{1}{r}\sum_{i=0}^{3}\frac{\partial V_i}{\partial\theta} = \frac{1}{4\pi\varepsilon_0}\sum_{i=0}^{3}\frac{q_i z_i\sin\theta}{\left[r^2 + z_i^2 - 2rz_i\cos\theta\right]^{3/2}}$$

4.16 (cont)

Charge density on the plane $E_z(r,\theta) = -E_\theta(r,\theta)$ when $\sin(\pi/2)=1, \cos(\pi/2)=0$

$$\sigma(r) = \varepsilon_0 E_z(r,\theta)\,|_{\theta=\pi/2} = \frac{-1}{4\pi}\sum_{i=0}^{3}\frac{q_i z_i}{\left[r^2+z_i^2\right]^{3/2}}$$

Total charge on the plane

$$Q_p = \int_0^{2\pi} d\phi \int_{R_0}^{\infty} \sigma(r) r dr = \frac{-1}{2}\sum_{i=0}^{3} q_i z_i \int_{R_0}^{\infty}\left[r^2+z_i^2\right]^{-3/2} r dr = \frac{-1}{2}\sum_{i=0}^{3} q_i z_i (R_0^2+z_i^2)^{-1/2}$$

Image charges: $q_1 = -q_0\dfrac{R_0}{z_0}; \quad z_1 = \dfrac{R_0^2}{z_0}$ $\quad q_2 = q_0\dfrac{R_0}{z_0}; \quad z_2 = -\dfrac{R_0^2}{z_0}$ $\quad q_3 = -q_0; \quad z_3 = -z_0$

$$Q_p = \frac{-1}{2}\left[2q_0 z_0 (R_0^2+z_0^2)^{-1/2} - 2q_0\left(R_0^3/z_0^2\right)(R_0^2+R_0^4/z_0^2)^{-1/2}\right]$$
$$= q_0\left(R_0^2-z_0^2\right)/\left[z_0(R_0^2+z_0^2)^{1/2}\right]$$

Using $q_1+q_2+q_3=-q_0$

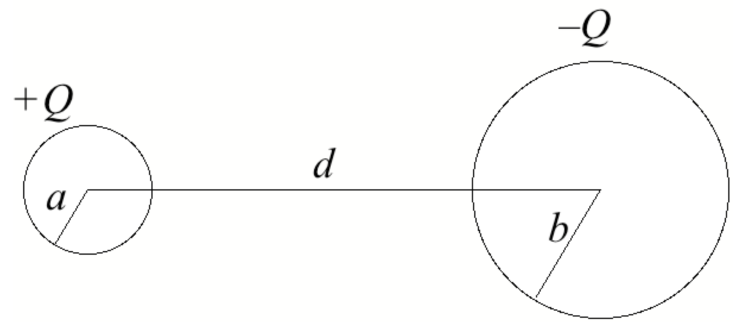
$$f_{hemi} = \left[(-q_0)-Q_p\right]/(-q_0) = 1-\left(z_0^2-R_0^2\right)/\left[z_0(R_0^2+z_0^2)^{1/2}\right]$$

4.18

<u>Two spheres charged by a battery</u>

Because the two spheres are very far apart the potentials of each sphere are independent.

Charged by a battery, each must have the same magnitude of charge



Reference each potential with respect to V = 0 at infinity

Comparing the potentials of each sphere with respect to V = 0 at infinity

$$V(r) = \frac{q}{4\pi\varepsilon_0 r}$$

$$V_0 = V(b) - V(a) = \frac{Q}{4\pi\varepsilon_0 b} - \frac{-Q}{4\pi\varepsilon_0 a}$$

$$= \frac{Q(a+b)}{4\pi\varepsilon_0 ab}$$

$$C = \frac{Q}{V_0} = 4\pi\varepsilon_0 \frac{ab}{(a+b)}$$

4.20

See lecture 16 for derivation of charge density on a grounded sphere in an external field

General potential spherical coordinates in simple situations

$$V(r,\theta) = \frac{A}{r} + B + \frac{C\cos\theta}{r^2} + Dr\cos\theta$$

Grounded sphere in external field

$$Q = 0 \quad \Rightarrow \quad A = 0$$

$$V(a,\theta) = 0; \quad V(r,\theta)\big|_{r\to\infty} = -E_0 z$$

$$\sigma(\theta) = 3\varepsilon_0 E_0 \cos\theta$$

New boundary conditions

$$Q = Q_0 \quad \Rightarrow \quad A \neq 0$$

$$V'(a,\theta) = \frac{Q_0}{4\pi\varepsilon_0 a}$$

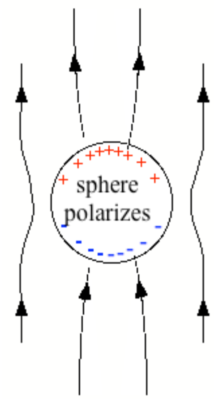
New term

$$V'(r,\theta) = \frac{A}{r} + \frac{E_0 a^3 \cos\theta}{r^2} - E_0 r\cos\theta$$

$$V'(a,\theta) = \frac{A}{a} + E_0 a\cos\theta - E_0 a\cos\theta = \frac{Q_0}{4\pi\varepsilon_0 a}$$

$$A = \frac{Q_0}{4\pi\varepsilon_0}$$

Add positive charge Q_0 here



$$E_r'(r,\theta) = -\frac{\partial V(r,\theta)}{\partial r} = \frac{Q_0}{4\pi\varepsilon_0 r^2} + \frac{2E_0 a^3\cos\theta}{r^3} + E_0\cos\theta$$

$$E_r'(a,\theta) = \frac{Q_0}{4\pi\varepsilon_0 a^2} + 3E_0\cos\theta$$

$$\sigma'(\theta) = \varepsilon_0 E_r(a,\theta) = \frac{Q_0}{4\pi a^2} + 3\varepsilon_0 E_0\cos\theta$$

$$\sigma'(\theta) > 0; \cos\theta = -1$$

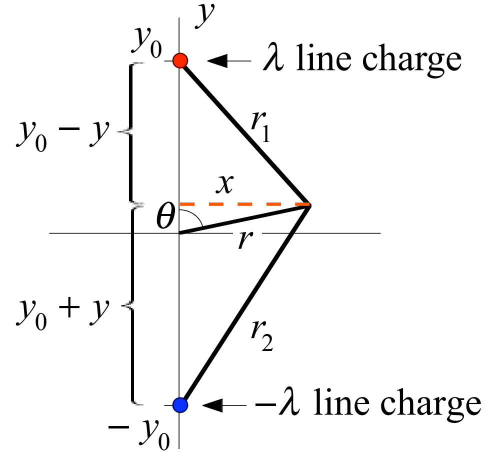
$$\frac{Q_0}{4\pi a^2} = 3\varepsilon_0 E_0$$

$$Q_0 = 12\pi a^2 \varepsilon_0 E_0$$

4.22

Line charge above a grounded conducting plane

Image with another line charge $-\lambda$ below the mid-plane



Single line charge potential

$$V(r) = \frac{-\lambda \ln r}{2\pi\varepsilon_0}$$

Double line charge potential

$$V(r) = \frac{-\lambda \ln\left(r_1/r_2\right)}{2\pi\varepsilon_0}$$

Electric field

$$E_y = -(\nabla V)_y = \frac{\lambda}{2\pi\varepsilon_0}\left[\frac{y-y_0}{r^2+y_0^2-2yy_0} - \frac{y+y_0}{r^2+y_0^2+2yy_0}\right]$$

Field on mid-plane

$$E_y(x) = \frac{-\lambda y_0}{\pi\varepsilon_0\left(x^2+y_0^2\right)}$$

Charge density on conductor

$$\sigma(x) = \varepsilon_0 E_y(x) = \frac{-\lambda y_0}{\pi\left(x^2+y_0^2\right)}$$

Linear charge density on conductor

$$\lambda_z = \int_{-\infty}^{\infty} \sigma(x)\,dx = \frac{-\lambda y_0}{\pi}\int_{-\infty}^{\infty}\frac{dx}{\left(x^2+y_0^2\right)} = \frac{-\lambda}{\pi}\left[\arctan\frac{x}{y_0}\right]_{-\infty}^{\infty}$$

$$\lambda_z = \frac{-\lambda}{\pi}\left[\arctan(\infty) - \arctan(-\infty)\right] = -\lambda$$