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# PHY481: Electromagnetism

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Solving Laplace's equation via "Separation of Variables"

- 1) Cartesian coordinates (review)
- 2) Spherical coordinates
- 3) Cylindrical coordinates

## 4.10a

Find potential of an upward facing dipole above a grounded conductor

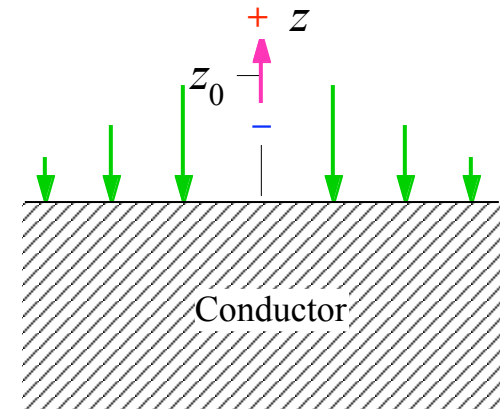
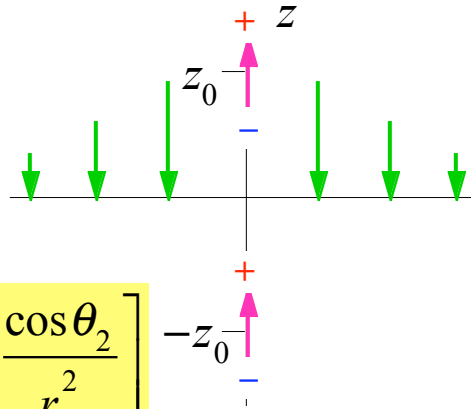
Potential via M.o.I is two dipoles (SAME DIRECTION)

Single dipole potential

$$V_d(r, \theta) = \frac{p \cos \theta}{4\pi\epsilon_0 r^2}$$

Double dipole potential

$$V_1(r, \theta) + V_2(r, \theta) = \frac{p}{4\pi\epsilon_0} \left[ \frac{\cos \theta_1}{r_1^2} + \frac{\cos \theta_2}{r_2^2} \right]$$



Double dipole potential on z-axis for  $z \gg z_0$

$$V_{2d}(z, 0) = \frac{p}{4\pi\epsilon_0} \left[ \frac{1}{(z - z_0)^2} + \frac{1}{(z + z_0)^2} \right] \quad \theta_1 = \theta_2 = 0$$

$$V_{2d}(z, 0) = \frac{p}{2\pi\epsilon_0 z^2}, \quad z \gg z_0$$

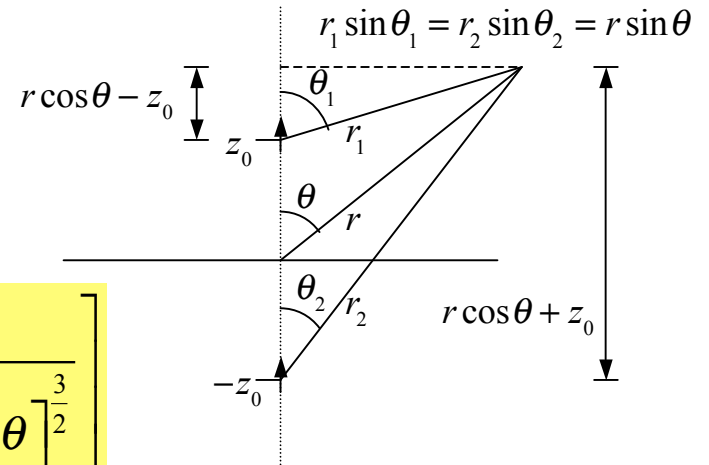
$$\mathbf{E} = -\Delta V = \frac{p}{\pi\epsilon_0 z^3} \hat{\mathbf{k}}$$

## 4.10a continued

$$V = V_1(r_1, \theta_1) + V_2(r_2, \theta_2) = \frac{p}{4\pi\epsilon_0} \left[ \frac{\cos\theta_1}{r_1^2} + \frac{\cos\theta_2}{r_2^2} \right]$$

$$r_1^2 = r^2 + z_0^2 - 2rz_0 \cos\theta \quad r_1 \cos\theta_1 = r \cos\theta - z_0$$

$$r_2^2 = r^2 + z_0^2 + 2rz_0 \cos\theta \quad r_2 \cos\theta_2 = r \cos\theta + z_0$$



$$V(r, \theta) = \frac{p}{4\pi\epsilon_0} \left[ \frac{r \cos\theta - z_0}{[r^2 + z_0^2 - 2rz_0 \cos\theta]^{\frac{3}{2}}} + \frac{r \cos\theta + z_0}{[r^2 + z_0^2 + 2rz_0 \cos\theta]^{\frac{3}{2}}} \right]$$

$$E_\theta(r, \theta) = -\frac{p}{4\pi\epsilon_0} \frac{1}{r} \frac{\partial V}{\partial \theta}$$

$$E_\theta(r, \pi/2) = \frac{p}{2\pi\epsilon_0 [r^2 + z_0^2]^{\frac{3}{2}}} \left[ 1 - \frac{3z_0^2}{[r^2 + z_0^2]} \right]; \quad \hat{\theta}(\theta = \pi/2) = -\hat{k}$$

$$\sigma = \epsilon_0 E_z = \frac{p}{2\pi (r^2 + z_0^2)^{3/2}} \left[ \frac{3z_0^2}{(r^2 + z_0^2)} - 1 \right]$$

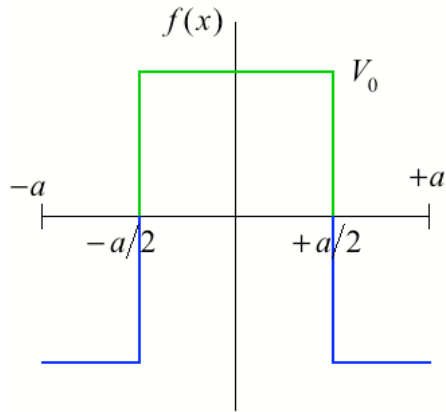
Where is charge density zero?

$$\sigma = 0 = \frac{3z_0^2}{(r^2 + z_0^2)} - 1; \quad r = \sqrt{2}z_0$$

Is total charge on surface = 0?

$$q = 2\pi \int_0^\infty \sigma r dr = p \int_0^\infty \frac{r dr}{(r^2 + z_0^2)^{3/2}} \left[ \frac{3z_0^2}{(r^2 + z_0^2)} - 1 \right] = p \left[ \frac{r^2}{(r^2 + z_0^2)^{3/2}} \right]_0^\infty = 0$$

# Coefficients in a Fourier Series



Find Fourier series expansion of  $f(x) = V_0$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi x}{a} + b_n \sin \frac{n\pi x}{a} \right]$$

$$a_m = \frac{1}{a} \int_{-a}^a f(x) \cos \frac{m\pi x}{a}$$

$$b_m = \frac{1}{a} \int_{-a}^a f(x) \sin \frac{m\pi x}{a} = 0$$

function  $f(x)$  is even  
so no sine terms

$$= \frac{1}{m\pi} \left[ -V_0 \sin \frac{m\pi x}{a} \Big|_{-a}^{-a/2} + V_0 \sin \frac{m\pi x}{a} \Big|_{-a/2}^{+a/2} - V_0 \sin \frac{m\pi x}{a} \Big|_{+a/2}^{+a} \right]$$

Be careful to break  
integral into pieces !!

$$a_m = \begin{cases} \frac{4V_0}{m\pi} (-1)^n, & m = 2n + 1 \text{ (odd)} \\ 0 & , m = 2n \text{ (even)} \end{cases}$$

$$f(x) = \frac{4V_0}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos \left[ (2n+1) \frac{\pi x}{a} \right]$$

# Coordinate independence in separable solutions

Cartesian coordinate separable solutions  $V(\mathbf{x}) = X(x)Y(y)Z(z)$

Translational independence  $x \rightarrow x'$   $V(\mathbf{x}') = X(x')Y(y)Z(z)$

$$\nabla^2 V / V = \frac{1}{X(x')} \frac{d^2 X(x')}{dx'^2} + \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} + \frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} = 0$$

Values of the last two terms (Y and Z terms) have not changed.

Yet the sum is still ZERO. Each of the 3 terms must be a constant.

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = k_1^2$$

$$\frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = k_2^2$$

$$\frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} = k_3^2$$

Solution to Laplace's equation must have

$$k_1^2 + k_2^2 + k_3^2 = 0$$

# Character of solutions

$$k_1^2 > 0$$

$$X(x) = Ae^{k_1 x} + Be^{-k_1 x}$$

Allowable values of  $A$ ,  $B$ , and  $k_1$  determined by symmetries and boundary conditions

If region unbounded +x

$$A = 0, \quad X(x) = Be^{-k_1 x}$$

If region unbounded -x

$$B = 0, \quad X(x) = Ae^{k_1 x}$$

If region bounded + & -

$$X(x) = A' \cosh k_1 x + B' \sinh k_1 x$$

Even symmetry

Odd symmetry

$$\cosh k_1 x = \left( \frac{e^{k_1 x} + e^{-k_1 x}}{2} \right); \quad \sinh k_1 x = \left( \frac{e^{k_1 x} - e^{-k_1 x}}{2} \right)$$

$$k_1^2 < 0$$

$$X(x) = Ae^{ik_1' x} + Be^{-ik_1' x}$$

Suggests using Fourier series for  $A$ ,  $B$ ,  $k_1$

Even symmetry

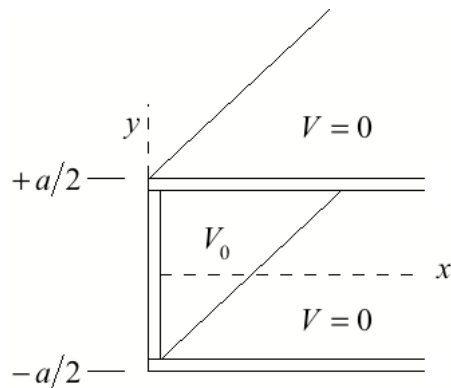
Odd symmetry

$$X(x) = A' \cos k_1' x + B' \sin k_1' x$$

$$\cos k_1 x = \left( \frac{e^{ik_1 x} + e^{-ik_1 x}}{2} \right); \quad \sin k_1 x = \left( \frac{e^{ik_1 x} - e^{-ik_1 x}}{2i} \right)$$

# Example: Long narrow channel

Large top and bottom grounded plates, potential  $V_0$  on the end plate  
Find potential everywhere between the plates.



Even y symmetry

$$V(x, y) = X(x)Y(y)$$

problems without z dependence  
all start the same way

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = 0$$

Laplace's equation

Each term must be a constant

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = k_1^2$$

$$\frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = k_2^2$$

$$k_1^2 + k_2^2 = 0$$

$$k_1^2 = +k^2, \quad k_2^2 = -k^2$$

First determine Y

$$\frac{d^2 Y(y)}{dy^2} = -k^2 Y(y)$$

$$Y(y) = A \cos ky + B \sin ky$$

$$Y(y) = A \cos ky \quad B = 0$$

V = 0 on top and bottom plate

$$Y(a/2) = \cos(ka/2) = 0$$

$$k = (2n+1) \frac{\pi}{a}; \quad n = 0, 1, 2, \dots$$

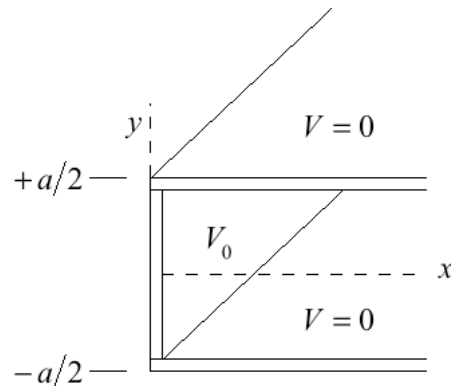
Solution with coefficients

$$Y(y) = \sum_{n=0}^{\infty} A_n \cos \left[ (2n+1) \frac{\pi y}{a} \right]$$

Any one of these values for k, satisfies the boundary condition at top and bottom, But an infinite series is needed to satisfy  $V(0, y) = V_0$

## Example (cont'd)

Determine the X function



$$\frac{d^2 X(x)}{dx^2} = +k^2 X(x)$$

General  
solution

$$X(x) = Ae^{kx} + Be^{-kx}$$

Finite at large +x

$$X(x \rightarrow \infty) = 0, \quad A = 0$$

$$X(x) = Be^{-kx}$$

From Y  
solution

$$k = (2n+1)\frac{\pi}{a}; \quad n = 0, 1, 2, \dots$$

$$X(x) = Be^{-(2n+1)\pi x/a}$$

V = XY and determine the coefficients

$$C_n = A_n B$$

$$V(x, y) = X(x)Y(y) = \sum_{n=0}^{\infty} C_n \cos \frac{(2n+1)\pi y}{a} e^{-(2n+1)\pi x/a}$$

C<sub>n</sub> by Fourier Integral

On left boundary, Potential is V<sub>0</sub>

Be careful to break  
integral into pieces !!

$$f(y) = \begin{cases} V_0 & -a/2 < y < a/2 \\ -V_0 & \text{elsewhere} \end{cases}$$

$$V(0, y) = V_0 = \sum_{n=0}^{\infty} C_n \cos \left[ \frac{(2n+1)\pi y}{a} \right]$$

$$C_n = \frac{1}{a} \left[ \int_{-a}^{+a} f(y) \cos \frac{(2n+1)\pi y}{a} dy \right] = \frac{4V_0 (-1)^n}{(2n+1)\pi}$$

Finally a solution

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos \frac{(2n+1)\pi y}{a} e^{-(2n+1)\pi x/a}$$

# Separation of Variables - spherical coordinates

Laplace's equation for potential in Spherical coordinates (no phi dep.)

$$\nabla^2 V(r, \theta) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right)$$

Solutions are often of the "separable" type,

$$V(r, \theta) = R(r)\Theta(\theta) \quad \text{or linear combinations of such solutions:}$$

$$V(\mathbf{x}) = \sum_{\ell=0}^{\infty} V_{\ell}(\mathbf{x});$$

$$V_{\ell}(\mathbf{x}) = R_{\ell}(r)\Theta_{\ell}(\theta)$$

Inserting this form into Laplace's equation and multiplying by  $r^2/V$

this depends only on  $r$    this depends only on  $\theta$

$$\frac{r^2}{V} \nabla^2 V(r, \theta) = \frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = 0$$

Terms must be constants    $+\ell(\ell+1)$     $-\ell(\ell+1)$    that sum to zero

$$\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - \ell(\ell+1) R = 0$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \ell(\ell+1) \Theta = 0$$

# Solving for functions $R(r)$ and $\Theta(\theta)$

$V(r, \theta) = R(r)\Theta(\theta)$  Separation of variables, solving Laplace's equation

Equation for  $R$ ,

$$\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - \ell(\ell + 1) R = 0$$

General solution

$$R(r) = Ar^\ell + \frac{B}{r^{\ell+1}}$$

non believers check this

$$\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = \ell(\ell + 1) R$$

and equation for  $\Theta$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \ell(\ell + 1) \Theta = 0$$

Legendre's equation for  $P(\cos \theta)$

$$\frac{d}{du} \left[ (1 - u^2) \frac{dP}{du} \right] + \ell(\ell + 1) P = 0$$

Let  $u = \cos \theta$ , and  $\Theta(\theta) = P(u)$

$$\frac{d\Theta}{d\theta} = \frac{du}{d\theta} \frac{dP}{du} = -\sin \theta \frac{dP}{du} = -(1 - u^2)^{\frac{1}{2}} \frac{dP}{du}$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = \frac{d}{du} \left( (1 - u^2) \frac{dP}{du} \right)$$

First 4 Legendre Polynomials

$P_\ell(\cos \theta)$

$$P_0(\cos \theta) = 1$$

$$P_1(\cos \theta) = \cos \theta$$

$$P_2(\cos \theta) = (3\cos^2 \theta - 1)/2$$

$$P_3(\cos \theta) = (5\cos^3 \theta - 3\cos \theta)/2$$

Solutions are Legendre Polynomials:

Finally,

$$V(r, \theta) = \sum_{\ell=0}^{\infty} \left( A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos \theta)$$

# Familiar problem in spherical coordinates

Orthogonality of Legendre polynomials:

$$\int_{-1}^1 P_m(\cos\theta) P_n(\cos\theta) d(\cos\theta) = \frac{2\delta_{mn}}{2n+1}$$

Consider a grounded conducting sphere radius,  $a$ , in a constant external field,  $E_0$  in  $z$  direction

Boundary condition  $V = 0$  on sphere

$$V(a, \theta) = 0 = \sum_{\ell=0}^{\infty} (A_{\ell} a^{\ell} + B_{\ell} a^{-(\ell+1)}) P_{\ell}(\cos\theta)$$

Multiply by  $P_n(\cos\theta)$  and integrate

$$\int_{-1}^1 V(a, \theta) P_n(\cos\theta) d(\cos\theta) = 0 = \sum_{\ell=0}^{\infty} \left( A_{\ell} a^{\ell} + \frac{B_{\ell}}{a^{\ell+1}} \right) \int_{-1}^1 P_{\ell}(\cos\theta) P_n(\cos\theta) d(\cos\theta)$$

$$0 = (A_n a^n + B_n a^{-(n+1)}) 2/(2n+1) \quad B_n = -A_n a^{2n+1}$$

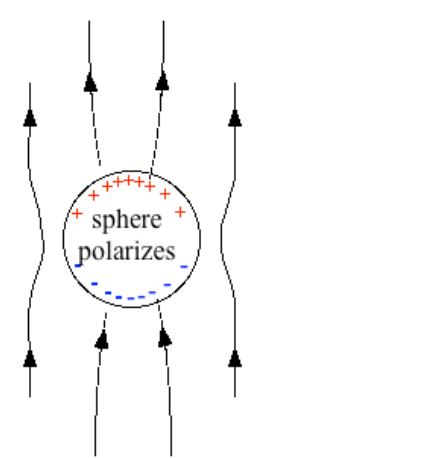
Boundary condition

$$V(r \rightarrow \infty, \theta) = -E_0 z = -E_0 r \cos\theta$$

$$\cos\theta = P_1(\cos\theta) \Rightarrow \ell = 1 \quad A_1 = -E_0$$

Same as solution found in Lecture 16

$$V(r, \theta) = \left( -E_0 r + \frac{E_0 a^3}{r^2} \right) \cos\theta$$



# Cylindrical coordinates

Laplace's equation in cylindrical coordinates

$$\nabla^2 V(r, \theta) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} = 0$$

Separation of variables

$$V(r, \phi) = R(r) \Phi(\phi)$$

r dependence    $\phi$  dependence

$$\frac{r^2}{V} \nabla^2 V(r, \phi) = \frac{r}{R} \frac{\partial}{\partial r} \left( r \frac{\partial R}{\partial r} \right) + \frac{1}{\Phi} \frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

Separate solutions

$$\Phi_n(\phi) = C_n \cos n\phi + D_n \sin n\phi$$

Constants sum to zero

$$n^2$$

$$-n^2$$

$$R_n(r) = A_n r^n + B_n r^{-n}$$

$$R_0(r) = A \ln r + B$$

Solution with series to insure boundary conditions can be satisfied

$$V(r, \phi) = A \ln r + B + \sum_{n=1}^{\infty} \left( A_n r^n + B_n r^{-n} \right) \left( C_n \cos n\phi + D_n \sin n\phi \right)$$

# Cylinder problem

Half cylinders, radius  $R$ .  $V = +V_0$  on right half, and  $V = -V_0$  on left half  
Find potential inside and out.

$$V(r, \phi) = A \ln r + B + \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) (C_n \cos n\phi + D_n \sin n\phi)$$

Boundary conditions at  $r = 0$  and  $r = \infty$

$$V_{Int}(r, \phi) = \sum_{n=1}^{\infty} A_n r^n \cos n\phi$$

$$V_{Ext}(r, \phi) = \sum_{n=1}^{\infty} B_n r^{-n} \cos n\phi$$

$$V_{Int}(R, \phi) = V_{Ext}(R, \phi)$$

$$A_n R^n = B_n R^{-n} = c_n$$

$$A_n = \frac{c_n}{R^n}; \quad B_n = c_n R^n$$

Use Orthogonality

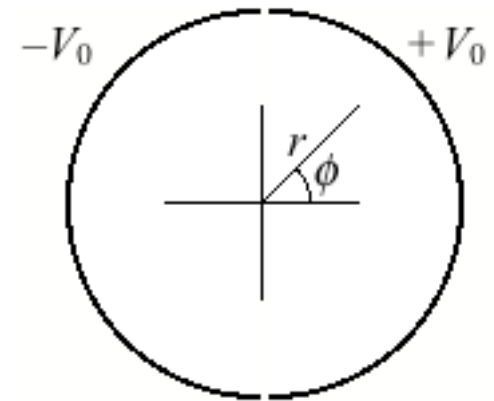
Be careful to break Fourier  
integral into pieces !!

$$V_{Int}(r, \phi) = \sum_{n=1}^{\infty} c_n \frac{r^n}{R^n} \cos n\phi$$

$$V_{Ext}(r, \phi) = \sum_{n=1}^{\infty} c_n \frac{R^n}{r^n} \cos n\phi$$

$$f(\phi) = \begin{cases} V_0 & 1^{\text{st}} \& 4^{\text{rd}} \text{ quadrants} \\ -V_0 & 2^{\text{nd}} \& 3^{\text{th}} \text{ quadrants} \end{cases}$$

$$c_j = \frac{4V_0(-1)^j}{\pi(2j+1)} \sin \frac{(2j+1)\pi}{2}$$



Orthogonality

$$\int_0^{2\pi} \cos n\phi \cos m\phi = \pi \delta_{nm}$$

Complete solution

$$V_{Int}(r, \phi) = \frac{4V_0}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \frac{r^{2j+1}}{R^{2j+1}} \cos(2j+1)\phi$$

$$V_{Ext}(r, \phi) = \frac{4V_0}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \frac{R^{2j+1}}{r^{2j+1}} \cos(2j+1)\phi$$