PHY481: Electromagnetism

Solving Laplace's equation via "Separation of Variables"

1) Cartesian coordinates
2) Spherical coordinates
3) Cylindrical coordinates

Examples

Cartesian coordinates

- Determine nature of solutions on the rectangle (unbounded z)
  - Grounded at $x = -a/2$ or $+a/2$ then $X_k(x) = A_k\cos(kx) + B_k\sin(kx)$
  - Grounded at $y = -a/2$ or $+a/2$ then $Y_k(y) = A_k\cos(ky) + B_k\sin(ky)$
  - Other solution is then $U_k(u) = C_k\cosh(ku) + D_k\sinh(ku)$ (same k)
- Use periodic (sine or cosine) solution and boundary conditions to determine legal values of $k = n\pi x/a$ (same for both solutions)
- Generate general solution $V(x,y) = \Sigma X_k(x)Y_k(y)$
- Use additional boundary conditions to determine coeficients
  - Periodic B.C.: $V_b(x_0,y) = \Sigma X_k(x_0)Y_k(y)$ or $V_b(x,y_0) = \Sigma X_k(x)Y_k(y_0)$
  - Apply Fourier Integral to both sides: $\frac{1}{a}\int_{-a}^{a} V_b(x,y_0)\cos m\pi x/a$ (or y version)

    Be sure to split Fourier integral (-a,-a/2,a/2,a)
  - Use orthogonality on right side: $\int_{-a}^{a}\cos(n\pi x/a)\cos(m\pi x/a) = \delta_{mn}$ (or y version)
  - Determine coef. A_k, B_k, C_k, D_k
- Substitute into $V(x,y) = \Sigma X_k(x)_kY(y)$

From previous Exam 2

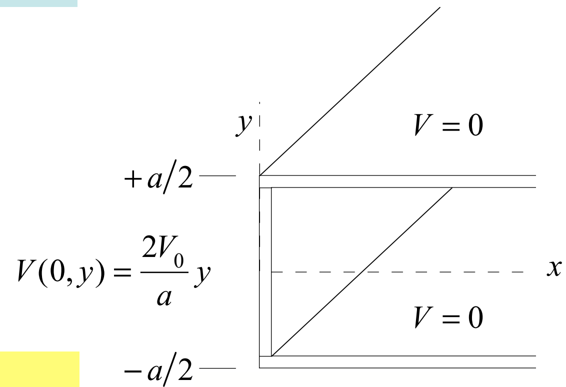
Boundary conditions $V(x,\pm a/2)=0,\ x\neq 0$ $\quad V(0,y)=2V_0 y/a$ $\quad V(x,y)\to 0,\ \text{as } x\to\infty$

Y Solution $Y(y)=A\cos\left(k\pi y/a\right)+B\sin\left(k\pi y/a\right)$

Apply boundary condition on Y solution

$$V(0,y)=2V_0\,y/a,\quad \text{an odd function} \Rightarrow\ A=0.$$

$$Y(\pm a/2)=B\sin\left(\pm k\pi/2\right)=0\qquad \Rightarrow\ k=2n,\ \text{an even integer.}$$

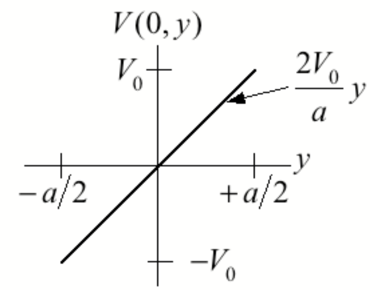


Apply boundary condition on X solution (same *k*)

$$X(x)=Ce^{2n\pi x/a}+De^{-2n\pi x/a};\quad V(x,y)\to 0,\ \text{as}\ x\to\infty,\quad \Rightarrow\ C=0$$

Most general solution consistent with boundary conditions

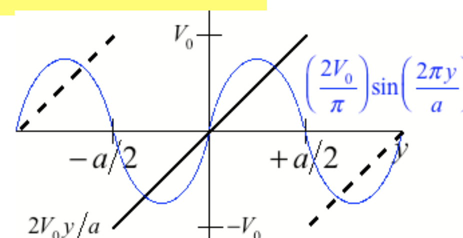
$$V(x,y)=\sum_{n=1}^{\infty}K_n\sin\left(2n\pi\, y/a\right)e^{-2n\pi x/a}\qquad V(0,y)=\sum_{n=1}^{\infty}K_n\sin\left(2n\pi\, y/a\right)$$



Determine coefficients

$$K_n=\frac{1}{a}\int_{-a}^{a}V(0,y)\sin\left(2n\pi\, y/a\right)dy=4\frac{2V_0}{a^2}\int_0^{a/2}y\sin\left(2n\pi\, y/a\right)dy=\frac{2V_0}{\pi}\left\{\frac{(-1)^{n+1}}{n}\right\}$$

Solution $$V(x,y)=\frac{2V_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(2n\pi y/a\right)e^{-2n\pi x/a}$$



Spherical coordinates: $V(r,\theta) = R(r)\Theta(\theta)$

Laplace's equation for potential in Spherical coordinates (no phi dep.)

$$\nabla^2 V(r,\theta) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)$$

Sep. of variables

$$V(r,\theta) = R(r)\Theta(\theta)$$

Separate solutions

$$R(r) = Ar^{\ell} + \frac{B}{r^{\ell+1}}$$

$$\Theta(\theta) \to F(\cos\theta) = P_\ell(\cos\theta)$$

Legendre Polynomials

$$P_0(\cos\theta) = 1$$

$$P_1(\cos\theta) = \cos\theta$$

$$P_2(\cos\theta) = \left(3\cos^2\theta - 1\right)/2$$

$$P_3(\cos\theta) = \left(5\cos^3\theta - 3\cos\theta\right)/2$$

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^{\ell} + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

Boundary condition on a sphere

$$V(a,\theta) = f(\theta)$$

$$\int_{-1}^{1} V(a,\theta)P_n(\cos\theta)\,d(\cos\theta)$$

Orthogonality

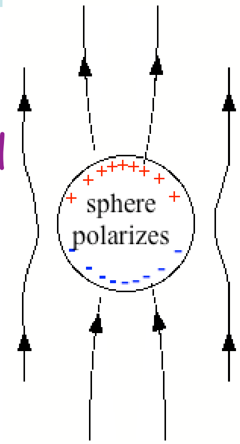
$$\int_{-1}^{1} P_m(\cos\theta)P_n(\cos\theta)\,d(\cos\theta) = \frac{2\delta_{mn}}{2n+1}$$

Familiar problem in spherical coordinates

Orthogonality of Legendre polynomials:

$$\int_{-1}^{1} P_m(\cos\theta)P_n(\cos\theta)\,d(\cos\theta) = \frac{2\delta_{\mathrm{mn}}}{2\mathrm{n}+1}$$

Consider a grounded conducting sphere radius, *a*, in a constant external field, E_0 in z direction



Boundary condition V = 0 on sphere

$$V(a,\theta) = 0 = \sum_{\ell=0}^{\infty}\left(A_\ell a^\ell + B_\ell a^{-(\ell+1)}\right)P_\ell(\cos\theta)$$

Multiply by $P_n(\cos\theta)$ and integrate

$$\int_{-1}^{1} V(a,\theta)P_n(\cos\theta)d(\cos\theta) = 0 = \sum_{\ell=0}^{\infty}\left(A_\ell a^\ell + \frac{B_\ell}{a^{\ell+1}}\right)\int_{-1}^{1} P_\ell(\cos\theta)P_n(\cos\theta)d(\cos\theta)$$

$$0 = \left(A_n a^n + B_n a^{-(n+1)}\right)2/(2n+1) \qquad B_n = -A_n a^{2n+1}$$

Boundary condition

$$V(r\to\infty,\theta) = -E_0 z = -E_0 r\cos\theta$$

$$\cos\theta = P_1(\cos\theta) \quad \boxed{\Longrightarrow \ell = 1} \quad A_1 = -E_0$$

uniform field   dipole   dipole moment

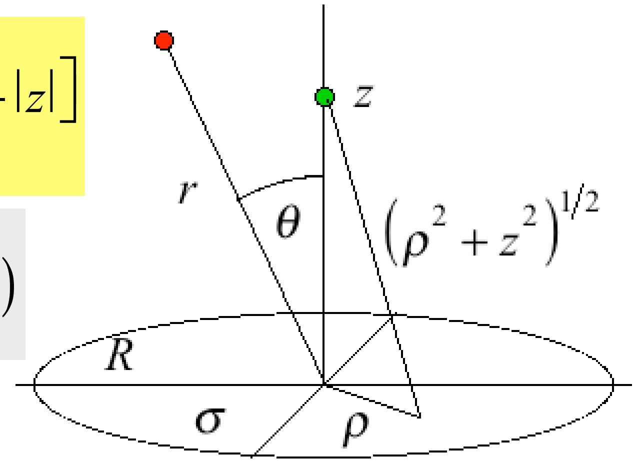
$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta) \longrightarrow = -E_0 r\cos\theta + \frac{E_0 a^3\cos\theta}{r^2}$$

$$E_0 a^3 = \frac{p}{4\pi\varepsilon_0}$$

Another spherical coordinate problem

Disk, charge density σ, radius R. Determine potential on z-axis, then at all angles.

$$V(r,0)=V(z,0)=\frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi}d\phi\int_0^R\frac{\rho\,d\rho}{\left(z^2+\rho^2\right)^{1/2}}=\frac{\sigma}{2\varepsilon_0}\left[(z^2+R^2)^{1/2}-|z|\right]$$



Use an expansion of it to find potential everywhere.

$$V(r,\theta)=\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

$$(z^2+R^2)^{1/2}=z\left(1+R^2/z^2\right)^{1/2}=z\left(1+R^2/2z^2-R^4/8z^4+\ldots\right)$$

$$f(\varepsilon)=1+f'(0)\varepsilon+f''(0)\varepsilon^2/2$$

$$f=(1+\varepsilon)^{1/2}\quad;f(0)=1$$
$$f'=(1+\varepsilon)^{-1/2}/2\quad;f'(0)=1/2$$
$$f''=-(1+\varepsilon)^{-1/2}/4\,;f''(0)=-1/4$$

$$(1+\varepsilon)^{1/2}=1+\varepsilon/2-\varepsilon^2/8$$

$$(1+x^2)^{1/2}=1+x^2/2-x^4/8+\ldots$$

$$V(z,0)=\frac{\sigma}{2\varepsilon_0}\left[(z^2+R^2)^{1/2}-|z|\right]=\frac{\sigma R^2}{2\varepsilon_0}\left[\frac{1}{2z}-\frac{R^2}{8z^3}+\ldots\right]$$

Match coefficients

All $A_\ell=0$

$$V(r,\theta)=\frac{\sigma R^2}{2\varepsilon_0}\left\{\left(\frac{B_0}{r^1}\right)P_0(\cos\theta)+\left(\frac{B_2}{r^3}\right)P_2(\cos\theta)\right\}$$

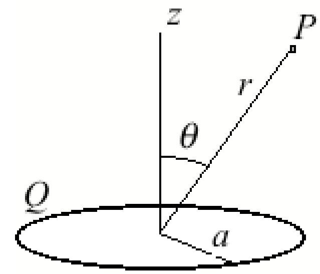
$$B_0=\frac{1}{2};B_2=-\frac{R^2}{8}$$

$$V(r,\theta)=\frac{\sigma\pi R^2}{4\pi\varepsilon_0}\left\{\left(\frac{1}{r}\right)-\left(\frac{R^2}{4r^3}\right)\left(\frac{3\cos\theta-1}{2}\right)+\ldots\right\}$$

From previous Exam 2: Ring of charge

Potential on z axis

$$V(z) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|} = \frac{Q}{4\pi\varepsilon_0}\left(z^2+a^2\right)^{-1/2} = \frac{Q}{4\pi\varepsilon_0 z}\left(1+a^2/z^2\right)^{-1/2}$$



Expansion in powers of a/z   let $\delta = \varepsilon^2 = a^2/z^2$

$$f(\delta) = (1+\delta)^{-1/2};\ f'(\delta) = -\tfrac{1}{2}(1+\delta)^{-3/2};\ f''(\delta) = +\tfrac{3}{4}(1+\delta)^{-5/2}$$

$$V(z) = \frac{Q}{4\pi\varepsilon_0 z}\left(f(0) + f'(0)\frac{a^2}{z^2} + f''(0)\frac{a^4}{2z^4} + \ldots\right)$$

$$= \frac{Q}{4\pi\varepsilon_0 z}\left(1 - \frac{a^2}{2z^2} + \frac{3a^4}{8z^4} + \ldots\right) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{z} - \frac{a^2}{2z^3} + \frac{3a^4}{8z^5} + \ldots\right)$$

Potential in spherical coordinates with boundary condition at large r

$$V(r\to\infty,\theta) = 0 \quad \Rightarrow A_\ell = 0 \qquad V(r,\theta) = \sum_{\ell=0}^{\infty} B_\ell r^{-(\ell+1)} P_\ell(\cos\theta)$$

Matching coefficients for solutions on the z-axis

$$V(z,0) = \sum_{\ell=0}^{\infty} B_\ell z^{-(\ell+1)} = \frac{B_0}{z} + \frac{B_2}{z^3} + \frac{B_4}{z^5} + \ldots = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{z} - \frac{a^2}{2z^3} + \frac{3a^4}{8z^5} + \ldots\right)$$

Ring of charge, continued

Matching coefficients for solutions on the z-axis

$$V(z,0)=\sum_{\ell=0}^{\infty}B_\ell z^{-(\ell+1)}=\frac{B_0}{z}+\frac{B_2}{z^3}+\frac{B_4}{z^5}+\ldots=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{z}-\frac{a^2}{2z^3}+\frac{3a^4}{8z^5}+\ldots\right)$$

No $\text{odd }\ell$ solutions

$$B_\ell=0\text{ for odd }\ell,\ B_0=\frac{Q}{4\pi\varepsilon_0};\quad B_2=\frac{Q}{4\pi\varepsilon_0}\frac{a^2}{2};\quad B_4=\frac{Q}{4\pi\varepsilon_0}\frac{3a^4}{8}$$

Insert coefficients into general solution

$$V(r,\theta)=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r}-\frac{a^2}{2r^3}P_2(\cos\theta)+\frac{3a^4}{8r^5}P_4(\cos\theta)+\ldots\right)$$

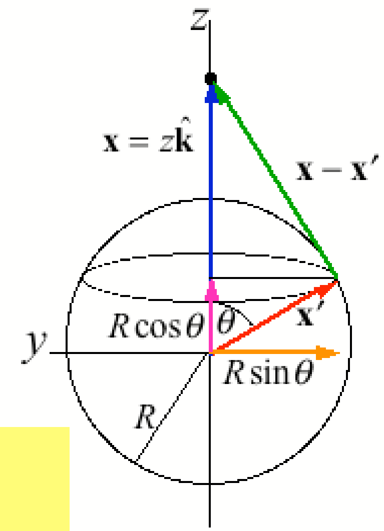
From previous Exam 2

Surface charge density $\sigma(\theta) = \sigma_0 \cos\theta$. Find Potential inside and out

$$V(z) = \int \frac{\rho(\mathbf{x}')\,d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$
$$= \frac{\sigma_0 R^2}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi \int_0^{\pi} \frac{\cos\theta\sin\theta d\theta}{\sqrt{R^2+z^2-2Rz\cos\theta}}$$
$$= \frac{\sigma_0 R^2}{2\varepsilon_0}\int_{-1}^{1} \frac{xdx}{\sqrt{R^2+z^2-2Rzx}}$$

$$\rho(\mathbf{x}')\,d^3x' = \sigma(\theta)\,d^2x'$$
$$= (\sigma_0\cos\theta)R^2\sin\theta d\theta d\phi$$

$$|\mathbf{x}-\mathbf{x}'| = \sqrt{(z-R\cos\theta)^2 + R^2\sin^2\theta}$$
$$= \sqrt{z^2+R^2-2Rz\cos\theta}$$



$$\int_{-1}^{1} \frac{xdx}{\sqrt{R^2+z^2-2Rzx}} = \frac{1}{3R^2z^2}\left[-\sqrt{R^2-2Rxz+z^2}\,(R^2+Rxz+z^2)\right]_{-1}^{1}$$
$$= \frac{1}{3R^2z^2}\left[-\sqrt{(R-z)^2}\,[R^2+Rz+z^2] + \sqrt{(R+z)^2}\,[R^2-Rz+z^2]\right]$$

$$\text{for } -R<z<R \quad = \{-(R-z)[R^2+Rz+z^2]+(R+z)[R^2-Rz+z^2]\}/3R^2z^2 = 2z/3R^2$$
$$\text{for } z>R \quad = \{-(z-R)[R^2+Rz+z^2]+(R+z)[R^2-Rz+z^2]\}/3R^2z^2 = 2R/3z^2$$
$$\text{for } z<-R \quad = \{(z-R)[R^2+Rz+z^2]-(R+z)[R^2-Rz+z^2]\}/3R^2z^2 = -2R/3z^2$$

$z<R: \quad V(z) = [\sigma_0/3\varepsilon_0]z\,;$ $\qquad z>R: \quad V(z) = [\sigma_0 R^3/3\varepsilon_0]z^{-2};$ $\qquad z<-R: \quad V(z) = -[\sigma_0 R^3/3\varepsilon_0]z^{-2}$

Cylindrical coordinates

Laplace's equation in cylindrical coordinates

$$\nabla^2 V(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Separation of variables

$$V(r,\phi) = R(r)\Phi(\phi)$$

<u>r dependence</u> <u>ϕ dependence</u>

$$\frac{r^2}{V}\nabla^2 V(r,\phi) = \frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

Constants sum to zero n^2 $-n^2$

Separate solutions

$$\Phi_n(\phi) = C_n \cos n\phi + D_n \sin n\phi$$

$$R_n(r) = A_n r^n + B_n r^{-n}$$

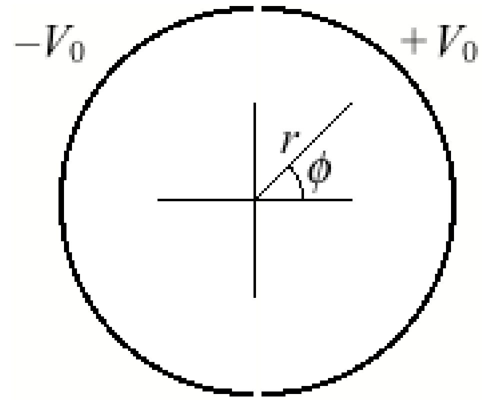
$$R_0(r) = A\ln r + B$$

Solution with series to insure boundary conditions can be satisfied

$$V(r,\phi) = A\ln r + B + \sum_{n=1}^{\infty}\left(A_n r^n + B_n r^{-n}\right)\left(C_n \cos n\phi + D_n \sin n\phi\right)$$

Cylinder problem

Half cylinders, radius R. $V = +V_0$ on right half, and $V = -V_0$ on left half
Find potential inside and out.



$$V(r,\phi) = A\ln r + B + \sum_{n=1}^{\infty}\left(A_n r^n + B_n r^{-n}\right)\left(C_n \cos n\phi + D_n \sin n\phi\right)$$

Boundary conditions at r = 0 and r = ∞

$$V_{Int}(r,\phi) = \sum_{n=1}^{\infty} A_n r^n \cos n\phi$$

$$V_{Ext}(r,\phi) = \sum_{n=1}^{\infty} B_n r^{-n} \cos n\phi$$

$$V_{Int}(R,\phi) = V_{Ext}(R,\phi)$$

$$A_n R^n = B_n R^{-n} = c_n$$

$$A_n = \frac{c_n}{R^n}; \quad B_n = c_n R^n$$

$$V_{Int}(r,\phi) = \sum_{n=1}^{\infty} c_n \frac{r^n}{R^n} \cos n\phi$$

$$V_{Ext}(r,\phi) = \sum_{n=1}^{\infty} c_n \frac{R^n}{r^n} \cos n\phi$$

Orthogonality

$$\int_0^{2\pi} \cos n\phi \cos m\phi = \pi\delta_{nm}$$

Use Orthogonality

Be careful to break Fourier integral into pieces !!

$$f(\phi) = \begin{cases} V_0 & 1^{\text{st}} \ \& \ 4^{\text{rd}} \text{ quadrants} \\ -V_0 & 2^{\text{nd}} \ \& \ 3^{\text{th}} \text{ quadrants} \end{cases}$$

$$\sin\left((2j+1)\pi/2\right) = (-1)^j$$

$$c_j = \frac{4V_0(-1)^j}{\pi(2j+1)}$$

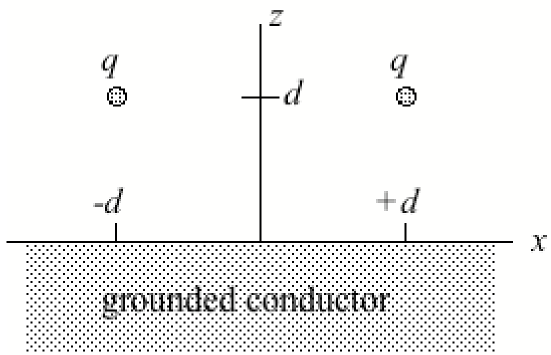
Complete solution

$$V_{Int}(r,\phi) = \frac{4V_0}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\frac{r^{2j+1}}{R^{2j+1}}\cos(2j+1)\phi$$

$$V_{Ext}(r,\phi) = \frac{4V_0}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\frac{R^{2j+1}}{r^{2j+1}}\cos(2j+1)\phi$$

From previous Exam 2 (Method of Images)

Two charges above grounded plane.
Determine potential and force on charge q.



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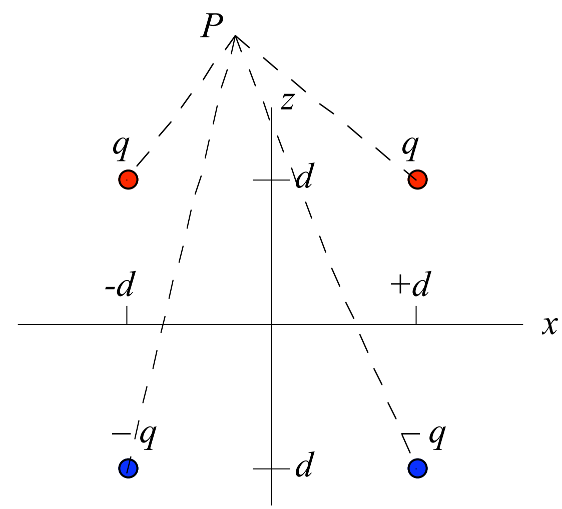
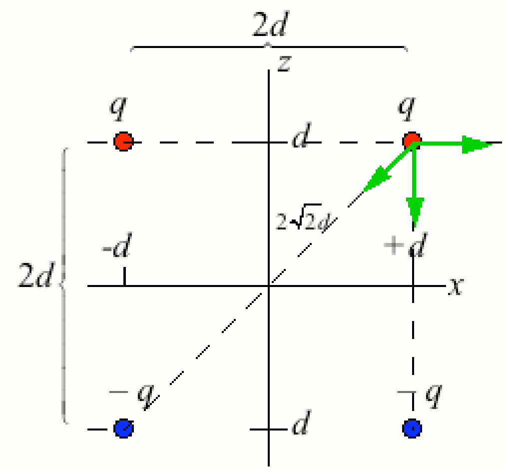


Potential above the plane

$$V(x,0,z)=\frac{q}{4\pi\varepsilon_0}\left\{\begin{array}{l}\dfrac{1}{\sqrt{(x-d)^2+(z-d)^2}}+\dfrac{1}{\sqrt{(x+d)^2+(z-d)^2}}\\[2ex] \quad -\dfrac{1}{\sqrt{(x-d)^2+(z+d)^2}}-\dfrac{1}{\sqrt{(x+d)^2+(z+d)^2}}\end{array}\right\}$$

$$\mathbf{F}=\frac{q^2}{4\pi\varepsilon_0}\left\{\frac{\hat{\mathbf{i}}}{4d^2}-\frac{\hat{\mathbf{k}}}{4d^2}-\frac{\hat{\mathbf{i}}+\hat{\mathbf{k}}}{8d^2\sqrt{2}}\right\}$$

$$=\frac{q^2}{16\pi\varepsilon_0 d^2}\left\{\left(1-\frac{\sqrt{2}}{4}\right)\hat{\mathbf{i}}-\left(1+\frac{\sqrt{2}}{4}\right)\hat{\mathbf{k}}\right\}$$



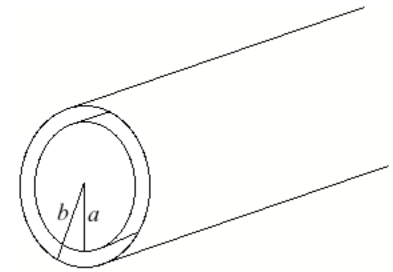
Capacitor problem from previous Exam 2

Cylindrical capacitor $d = b - a << a$ For V_a=V, determine C, Q, and U.

i) Means spacing very small

ii) Means cylinder areas are nearly the same

iii) Field between the cylinders must be ~ the field between parallel plates with the same area.



Initial state

$$C = \frac{\varepsilon_0 A}{d} = \frac{2\pi\varepsilon_0 aL}{b-a} \qquad Q = \frac{2\pi\varepsilon_0 aL}{b-a}V \qquad U = \frac{1}{2}\frac{Q^2}{C} = \frac{Q^2(b-a)}{4\pi\varepsilon_0 aL}$$

Force to pull inner cylinder out changing the overlap. $dL = L' - L$ (negative)

$$\frac{dU}{U} = -\frac{dL}{L} \qquad W = dU = -U\frac{dL}{L} \qquad \frac{dU}{dL} = -\frac{U}{L} \qquad F = -\frac{dU}{dL} = \frac{U}{L} = \frac{Q^2(b-a)}{4\pi\varepsilon_0 aL^2}$$

Without initial assumptions

$$V(r) = \frac{V_0 \ln(b/r)}{\ln(b/a)} \qquad E(r) = \frac{V_0}{\ln(b/a)r} \qquad E(a) = \frac{V_0}{\ln(b/a)a} = \frac{\sigma}{\varepsilon_0} = \frac{Q}{2\pi\varepsilon_0 aL}$$

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon_0 L}{\ln(b/a)} \approx \frac{2\pi\varepsilon_0 aL}{b-a}$$ because $\ln(b/a) = \ln[(a+d)/a] = \ln[1 + a/d] \approx (b-a)/a$