PHY481: Electromagnetism

HW5

5.3

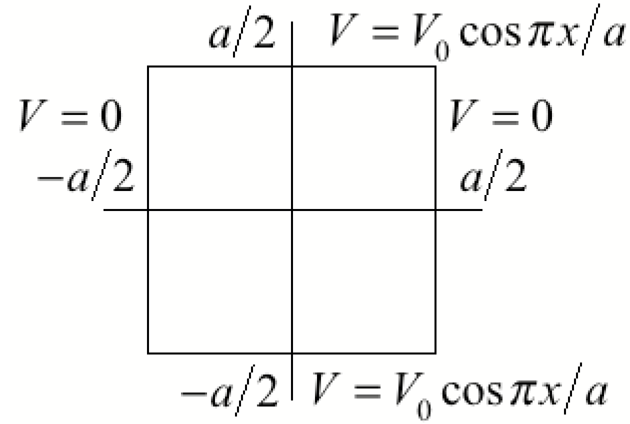
Boundary condition $V = V_0 \cos\left(\frac{\pi x}{a}\right)$ at $y = \pm a/2$

General solution (for even boundary conditions)

$$V(x,y) = \sum_{n=1}^{\infty} C_n \cos\left[(2n+1)\frac{\pi x}{a}\right] \cosh\left[(2n+1)\frac{\pi y}{a}\right]$$

$$V(x,\pm a/2) = V_0 \cos\left(\frac{\pi x}{a}\right)$$
$$= C_n \cos\left[(2n+1)\frac{\pi x}{a}\right] \cosh\left[(2n+1)\frac{\pi}{2}\right]$$

Square pipe cross section



Only non-zero term

$n = 0,\ C_0$ to be determined

Apply boundary condition to determine coefficient

$$V(x,\pm a/2) = C_0 \cos\left[\frac{\pi x}{a}\right] \cosh\left[\frac{\pi}{2}\right] = V_0 \cos\left[\frac{\pi x}{a}\right]$$

$$C_0 = \frac{V_0}{\cosh\left[\frac{\pi}{2}\right]}$$

Potential inside pipe

$$V(x,y) = \frac{V_0}{\cosh(\pi/2)} \cos(\pi x/a) \cosh(\pi y/a)$$

Phase factors

Prove: $A\sinh(x+\phi) = B\sinh(x) + C\cosh(x)$

$$A\left[\frac{e^{x+\phi} - e^{-(x+\phi)}}{2}\right] = B\left[\frac{e^{x} - e^{-x}}{2}\right] + C\left[\frac{e^{x} + e^{-x}}{2}\right]$$

$$A\left[\frac{e^{\phi}e^{x}}{2} - \frac{e^{-\phi}e^{-x}}{2}\right] = (C+B)\left[\frac{e^{x}}{2}\right] + (C-B)\left[\frac{e^{-x}}{2}\right]$$

$$C + B = Ae^{\phi}$$

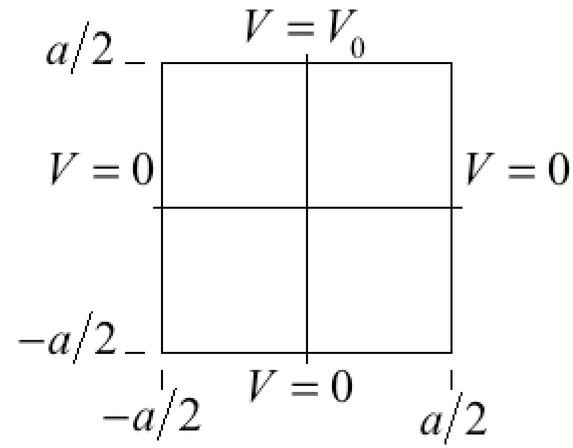
$$C - B = Ae^{-\phi}$$

$$C = A\frac{e^{\phi} - e^{-\phi}}{2}; \quad B = A\frac{e^{\phi} + e^{-\phi}}{2}$$

$$\sinh(x+\phi) = \cosh(\phi)\sinh(x) + \sinh(\phi)\cosh(x)$$

5.7

$$X(x)=B_n\cos[(2n+1)]\pi x/a$$ boundary conditions: $X(\pm a/2)=0$



$$Y(y)=C_n\sinh[(2n+1)\pi y/a]+D_n\cosh[(2n+1)\pi y/a]$$
$$=C_n'\sinh[(2n+1)\pi y/a+\phi_n]$$ instead use phase: ϕ_n

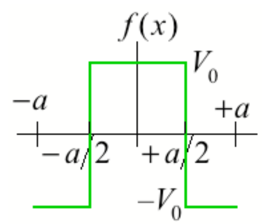
$$V(x,y)=\sum_{n=0}^{\infty}g_n\sinh[(2n+1)\pi\, y/a+\phi_n]\cos[(2n+1)\pi x/a]$$

$$V(x,-a/2)=0=\sum_{n=0}^{\infty}g_n\sinh[-(2n+1)\pi/2+\phi_n]\cos[(2n+1)\pi x/a]$$ bc: $Y(-a/2)=0$ sets phase

$$\phi_n=(2n+1)\pi/2$$

$$V(x,+a/2)=V_0=\sum_{n=0}^{\infty}g_n\sinh[(2n+1)\pi]\cos[(2n+1)\pi x/a]$$ bc: $Y(+a/2)=V_0$ sets coef. g_n

$$\frac{1}{a}\int_{-a}^{+a}f(x)\cos[(2n+1)\pi x/a]=g_n\sinh[(2n+1)\pi]$$



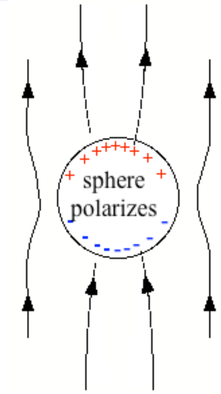
$$g_n=\frac{4V_0(-1)^n}{(2n+1)\pi}\frac{1}{\sinh[(2n+1)\pi]}$$

Be careful to break integral into pieces !!

$$V(x,y)=\frac{4V_0}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\frac{\sinh[(2n+1)\pi(y/a+1/2)]}{\sinh[(2n+1)\pi]}\cos[(2n+1)\pi x/a]$$

5.11

See lectures 18-20 for derivations of potential for a conducting sphere in an external field



$$V(r,\theta) = +\frac{E_0 a^3 \cos\theta}{r^2} - E_0 r \cos\theta$$

Dipole potential

$$V(r,\theta) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

Compare $1/r^2$ potentials

$$p = 4\pi\varepsilon_0 E_0 a^3$$

Dipole moment analogous to $p = qd$

Polarizability

$$p = \alpha E_0$$

$$\alpha = 4\pi\varepsilon_0 a^3$$

$$\mathbf{r} = a\cos\theta\,\hat{\mathbf{k}} + a\sin\theta\,\hat{\boldsymbol{\rho}}$$

$$\mathbf{p} = \int \sigma(\theta)\mathbf{r}\,dS = 2\pi a^3 \int (3\varepsilon_0 E_0 \cos\theta)\cos\theta\sin\theta\,d\theta\,\hat{\mathbf{k}}$$

$$\sigma(\theta) = \varepsilon_0 E_r(a,\theta) = -\varepsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=a} = 2\varepsilon_0 \frac{E_0 a^3 \cos\theta}{a^3} + \varepsilon_0 E_0 \cos\theta = 3\varepsilon_0 E_0 \cos\theta$$

$$p_z = 6\pi a^3 \varepsilon_0 E_0 \int_0^{\pi} \cos^2\theta \sin\theta\,d\theta = 6\pi a^3 \varepsilon_0 E_0 \int_1^{-1} -u^2\,du$$

Charge density

$$\sigma(\theta) = 3\varepsilon_0 E_0 \cos\theta$$

Dipole moment

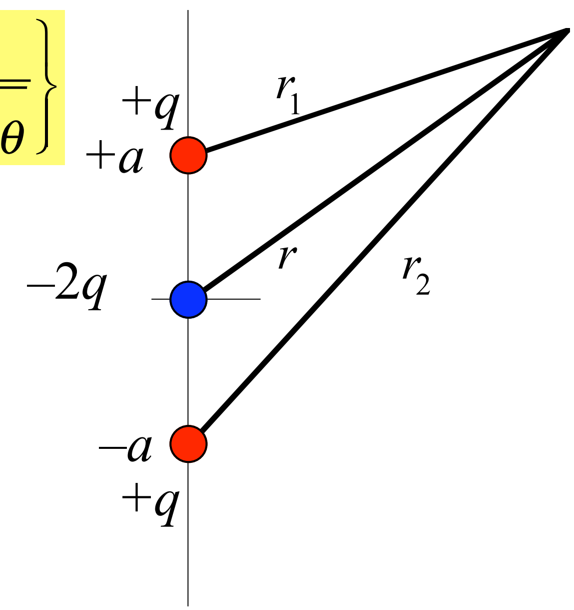
$$p_z = 4\pi a^3 \varepsilon_0 E_0$$

5.13

Linear quadrupole exact potential

$$V(r,\theta)=\frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}}+\frac{-2}{r}+\frac{1}{\sqrt{r^2+a^2+2ar\cos\theta}}\right\}$$

Linear quadrupole



Approximation

Expansions

$$\frac{1}{\sqrt{1+\varepsilon}}=1-\frac{1}{2}\varepsilon+\frac{3}{8}\varepsilon^2+\ldots$$

$$\frac{1}{\sqrt{r^2+a^2\pm 2ar\cos\theta}}=\frac{1}{r}\left\{1\pm\frac{a\cos\theta}{r}+\frac{3a^2\cos^2\theta-a^2}{2r^2}+\ldots\right\}$$

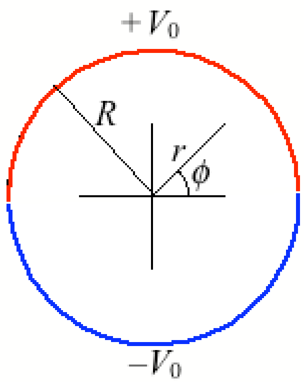
Monopole and dipole terms cancel

$$V(r,\theta)=\frac{qa^2}{4\pi\varepsilon_0}\left\{\frac{3\cos^2\theta-1}{2r^3}+\ldots\right\}=\frac{2qa^2}{4\pi\varepsilon_0}\frac{P_2(\cos\theta)}{r^3}+\ldots$$

5.15 (see lecture 19)

Half cylinders, radius R. $V = +V_0$ on upper half, and $V = -V_0$ on lower half
Find potential inside and out. (odd function of ϕ -> no cosine terms)

$$V(r,\phi) = A\ln r + B + \sum_{n=1}^{\infty}\left(A_n r^n + B_n r^{-n}\right)\left(C_n \cos n\phi + D_n \sin n\phi\right)$$



B.C. at r = 0 and r = ∞

$$V_{Int}(r,\phi) = \sum_{n=1}^{\infty} A_n r^n \sin n\phi$$

$$V_{Ext}(r,\phi) = \sum_{n=1}^{\infty} B_n r^{-n} \sin n\phi$$

B.C. at r = R

$$V_{Int}(R,\phi) = V_{Ext}(R,\phi)$$

$$A_n R^n = B_n R^{-n} = c_n \rightarrow A_n = \frac{c_n}{R^n}; \quad B_n = c_n R^n$$

Fourier Integral at r = R

Be careful to break Fourier integral into pieces !!

Use Orthogonality

$$\int_0^{2\pi} \cos n\phi \cos m\phi = \pi\delta_{nm}$$

$$\int_0^{2\pi} V(R,\phi)\sin m\phi\, d\phi = \sum_{n=1}^{\infty} c_n \int_0^{2\pi} \sin n\phi \sin m\phi\, d\phi$$

$$-V_0 \cos m\phi\Big|_0^{\pi} + V_0 \cos m\phi\Big|_{\pi}^{2\pi} = c_n \pi \delta_{mn}$$

$$c_m = \frac{4V_0}{\pi m} \quad m = 1,3,5\ldots$$

Complete solution

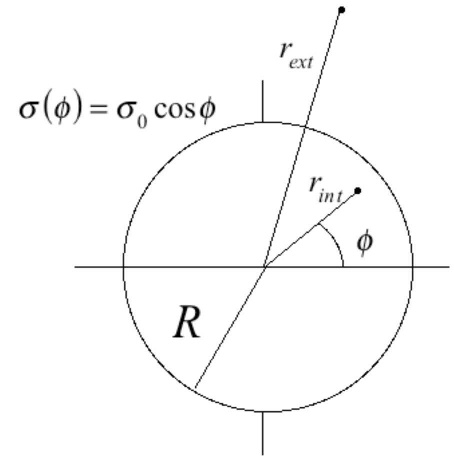
$$V_{Int}(r,\phi) = \frac{4V_0}{\pi}\sum_{m\text{ odd}}^{\infty}\frac{1}{m}\frac{r^m}{R^m}\sin m\phi$$

$$V_{Ext}(r,\phi) = \frac{4V_0}{\pi}\sum_{m\text{ odd}}^{\infty}\frac{1}{m}\frac{R^m}{r^m}\sin m\phi$$

5.16

General solution in cylindrical coordinates

$$V(r,\phi) = A\ln r + B + \sum_{n=1}^{\infty}\left(A_n r^n + B_n r^{-n}\right)\left(C_n \cos n\phi + D_n \sin n\phi\right)$$



$$E_r(R_{ext}) - E_r(R_{\text{int}}) = \frac{\sigma}{\varepsilon_0}$$

Boundary condition crossing the boundary

$$V(r_{\text{int}},\phi) = A_1 r\cos\phi \qquad E_n(r_{\text{int}},\phi) = -A_1\cos\phi$$

$$V(r_{ext},\phi) = \frac{B_1}{r}\cos\phi \qquad E_n(r_{ext},\phi) = +\frac{B_1\cos\phi}{r_{ext}^2}$$

Matching terms

Apply boundary conditions

$$\frac{B_1\cos\phi}{r_{ext}^2} + A_1\cos\phi = \frac{\sigma_0\cos\phi}{\varepsilon_0}$$

$$\frac{B_1}{R^2} + A_1 = \frac{\sigma_0}{\varepsilon_0}$$

$$V_{\text{int}}(R,\phi) = V_{ext}(R,\phi)$$

$$A_1 R\cos\phi = \frac{B_1}{R}\cos\phi$$

$$A_1 R = \frac{B_1}{R}$$

$$A_1 = \frac{\sigma_0}{2\varepsilon_0} \qquad B_1 = \frac{\sigma_0 R^2}{2\varepsilon_0}$$

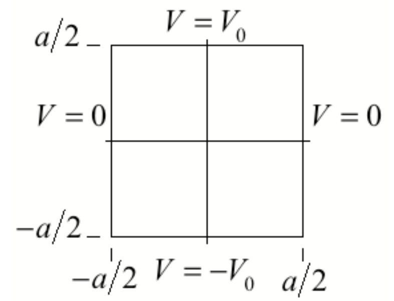
Solution inside and out

$$V(r_{\text{int}},\phi) = \frac{\sigma_0}{2\varepsilon_0} r\cos\phi; \quad V(r_{ext},\phi) = \frac{\sigma_0 R^2}{2\varepsilon_0 r}\cos\phi$$

$\color{blue}{5.28}$ (see problem 5.7)

$X(x) = B_n \cos[(2n+1)]\pi x/a$ boundary conditions: $X(\pm a/2) = 0$

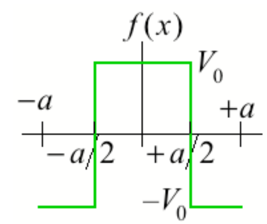
$Y(y) = C_n \sinh[(2n+1)\pi y/a]$ odd B. C. : $Y(\pm a/2) = \pm V_0$



$$V(x,y) = \sum_{n=0}^{\infty} g_n \sinh[(2n+1)\pi\, y/a] \cos[(2n+1)\pi x/a]$$ $$g_n = B_n C_n$$

$$V(x,+a/2) = V_0 = \sum_{n=0}^{\infty} g_n \sinh[(2n+1)\pi/2] \cos[(2n+1)\pi x/a]$$

B.C.: $Y(+a/2) = V_0$ sets coef. g_n



$$\frac{1}{a}\int_{-a}^{+a} f(x) \cos[(2n+1)\pi x/a]\, dx = g_n \sinh[(2n+1)\pi/2]$$

Be careful to break Fourier integral into pieces !!

$$\frac{V_0}{(2n+1)\pi}\left\{\begin{matrix} -\sin[(2n+1)\pi\, x/a]_{-a}^{-a/2} + \sin[(2n+1)\pi\, x/a]_{-a/2}^{a/2} \\ -\sin[(2n+1)\pi\, x/a]_{a/2}^{a} \end{matrix}\right\} = g_n \sinh[(2n+1)\pi/2]$$

$$g_n = \frac{4V_0(-1)^n}{(2n+1)\pi \sinh[(2n+1)\pi/2]}$$ $$V(x,y) = \frac{4V_0}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\frac{\sinh[(2n+1)\pi\, y/a]}{\sinh[(2n+1)\pi/2]}\cos[(2n+1)\pi x/a]$$

5.32

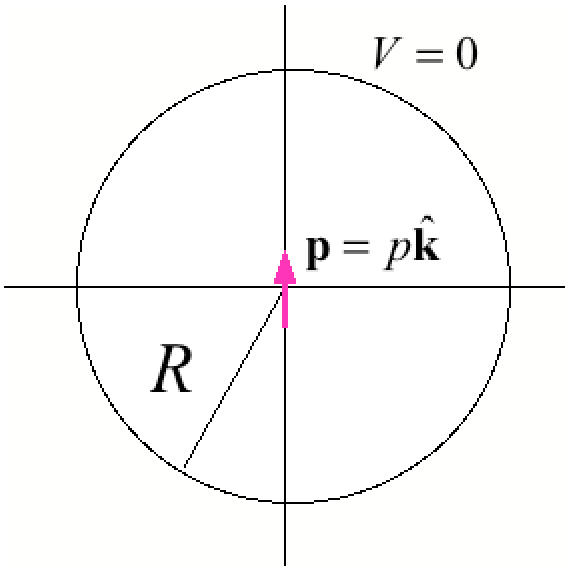
Dipole potential

$$V = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

General solution in spherical coordinates

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

Dipole in grounded conducting sphere



$$V(r,\theta) = \left(Ar + \frac{B}{r^2}\right)P_1(\cos\theta)$$

Only one term $\ell = 1$

Apply boundary conditions

r -> 0 $$V(r,\theta) \to \frac{B\cos\theta}{r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$ $$B = \frac{p}{4\pi\varepsilon_0}$$

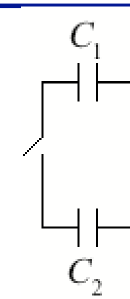
r = R $$V(R,\theta) = \left(AR + \frac{B}{R^2}\right) = 0$$

$$A = \frac{-B}{R^3} = \frac{-p}{4\pi\varepsilon_0 R^3}$$

Full Solution: $$V(r,\theta) = \frac{p}{4\pi\varepsilon_0}\left(\frac{-r}{R^3} + \frac{1}{r^2}\right)\cos\theta$$

2

Caps C_1 and C_2 carry charges Q_1 and Q_2 .
Close the switch - find new charges Q'_1 and Q'_2



Charge conserved: $Q_1' + Q_2' = Q_1 + Q_2$

Potentials become equal: $V' = Q_1'/C_1 = Q_2'/C_2; \quad Q_2' = (C_2/C_1)Q_1'$

$$Q_1' + (C_2/C_1)Q_1' = Q_1 + Q_2$$

$$Q_1' = \frac{Q_1 + Q_2}{[1 + (C_2/C_1)]}; \quad Q_2' = \frac{C_2}{C_1}\frac{Q_1 + Q_2}{(1 + (C_2/C_1))} = \frac{Q_1 + Q_2}{(1 + (C_1/C_2))}$$

Compare stored energy before and after switch is closed

$$Q_1 = Q_1' + \Delta Q; \quad Q_2 = Q_2' - \Delta Q$$

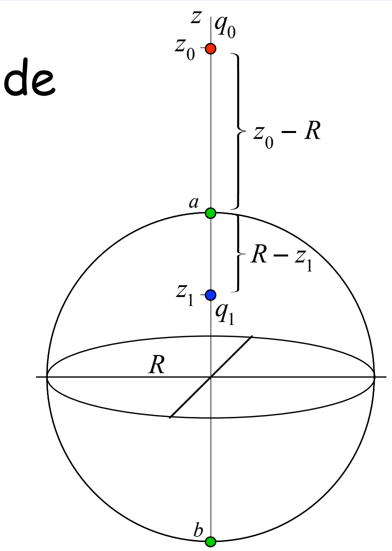
$U_1' + U_2'$ Potentials become equal: $V_1' - V_2' = 0$

$$U_1 + U_2 = \frac{1}{2}\left(\frac{Q_1^2}{C_1} + \frac{Q_2^2}{C_2}\right) = \frac{1}{2}\left(\frac{Q_1'^2}{C_1} + \frac{Q_2'^2}{C_2}\right) + \frac{1}{2}\left(\frac{(\Delta Q)^2}{C_1} + \frac{(\Delta Q)^2}{C_2}\right) + \Delta Q\left(\frac{Q_1'}{C_1} - \frac{Q_2'}{C_2}\right)$$

$$U_1 + U_2 = U_1' + U_2' + \frac{(\Delta Q)^2}{2C}; \qquad C = \frac{C_1 C_2}{C_1 + C_2}$$

3 a,b

Charge q_0 above a grounded conduction sphere
Determine a) image charge and position, b) potential outside



Potential at points (a) and (b) must be zero.

$$\frac{q_1}{R - z_1} = \frac{-q_0}{z_0 - R}; \quad q_1 = \frac{-q_0 (R - z_1)}{z_0 - R}$$

$$\frac{q_1}{R + z_1} = \frac{-q_0}{z_0 + R}; \quad q_1 = \frac{-q_0 (R + z_1)}{z_0 + R}$$

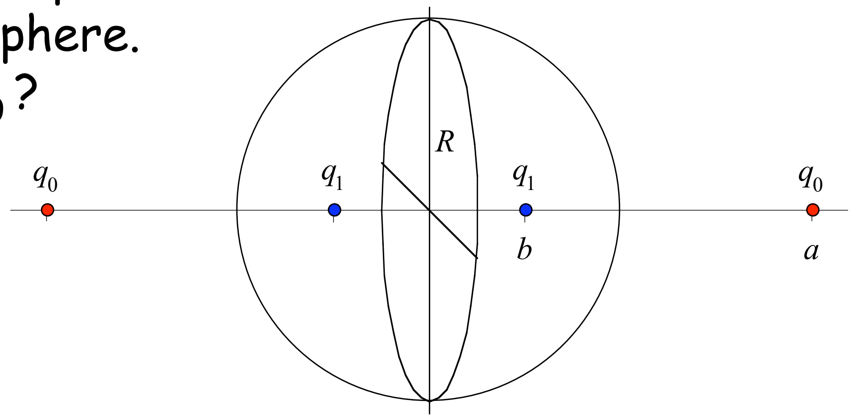
$$\frac{(R - z_1)}{z_0 - R} = \frac{(R + z_1)}{z_0 + R}; \qquad 2R^2 = 2z_1 z_0$$

$$a)\ z_1 = \underline{R^2 / z_0}; \quad q_1 = \underline{-q_0\, R / z_0}$$

$$b)\ V(r,\theta) = \underline{\frac{1}{4\pi\varepsilon_0}\left[\frac{q_0}{\sqrt{r^2 + z_0^2 - 2rz_0\cos\theta}} + \frac{q_1}{\sqrt{r^2 + z_1^2 - 2rz_1\cos\theta}}\right]}$$

3 c

Two charges q_0 outside grounded conduction sphere
Image: Two charges q_1 inside make V=0 on sphere.
What radius sphere makes force F = 0 on q_0 ?



$$a = z_0, b = R^2/a; \quad q_1 = -q_0\, R/a$$

$$F = \frac{q_0^2}{4a^2} + \frac{q_1 q_0}{(a+b)^2} + \frac{q_1 q_0}{(a-b)^2}$$
$$= \frac{q_0^2}{4a^2} - \frac{R}{a}\frac{q_0^2}{\left(a + R^2/a\right)^2} - \frac{R}{a}\frac{q_0^2}{\left(a - R^2/a\right)^2}$$

$$0 = \frac{1}{4a^2} - \frac{R}{a^3}\left(\frac{1}{\left(1 + R^2/a^2\right)^2} + \frac{1}{\left(1 - R^2/a^2\right)^2}\right)$$
$$= \frac{1}{4a^2} - \frac{R}{a^3}\left[\left(1 - 2\frac{R^2}{a^2} + 3\frac{R^4}{a^4} + \ldots\right) + \left(1 + 2\frac{R^2}{a^2} + 3\frac{R^4}{a^4} + \ldots\right)\right]$$

$$\frac{1}{(1 \pm x)^2} = 1 \mp 2x + 3x^2 + \ldots$$

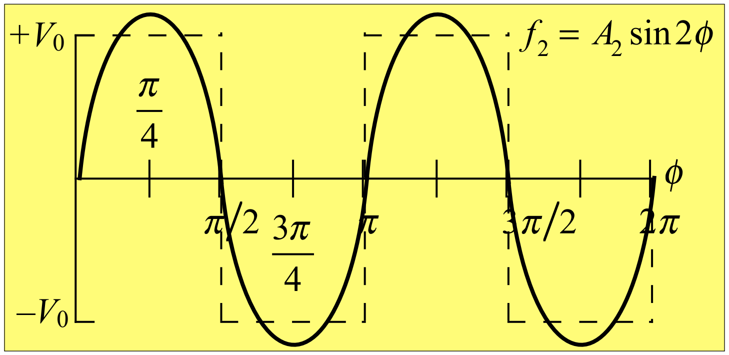
$$\frac{1}{4a^2} = \frac{2R}{a^3}\left[1 + 3\frac{R^4}{a^4} + \ldots\right]; \quad \frac{1}{4a^2} \approx \frac{2R}{a^3} \quad \text{assuming } R^2/a^2 << 1$$

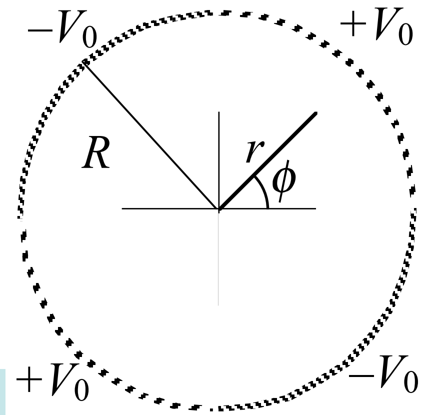
$$\underline{R \approx \frac{a}{8}}$$

4

Use this function: $f_n(\phi) = A_n \sin n\phi$

a) What is the best n to match this cylindrical potential?



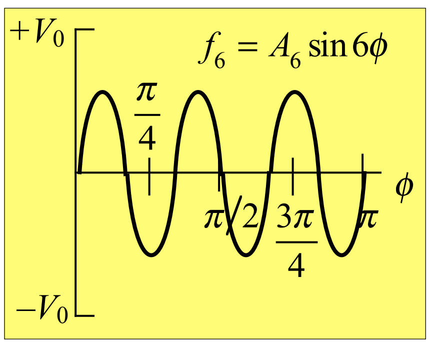


$$f_{n=2} = A_2 \sin 2\phi$$

zeros: $\pi/2,\ \pi,\ 3\pi/2,\ 2\pi,\ \ldots$

critical points: $\pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4,\ \ldots$

b) Next integer m where f_m includes these zeros and critical points?



zeros: $0,\ \pi/6,\ \pi/3,\ \underline{\pi/2},\ 2\pi/3,\ 5\pi/6,\ \underline{\pi},\ \ldots$

critical points: $\pi/12,\ \underline{\pi/4},\ 5\pi/12,\ 7\pi/12,\ \underline{3\pi/4},\ 3\pi/4,\ 11\pi/12,\ \ldots$

c) Find V(r,q) that satisfies Laplace's equation and compare first two terms with the functions above.

4 c

$$V(r,\phi)=\sum_{n=1}^{\infty}A_n r^{-n}\sin n\phi;\quad \text{Boundary: } V(R,\phi)=\sum_{n=1}^{\infty}A_n R^{-n}\sin n\phi$$

$$V(R,\phi)=\pm V_0$$

$$\frac{1}{\pi}\int_0^{2\pi}V(R,\phi)\sin m\phi\, d\phi=\frac{1}{\pi}\sum_{n=1}^{\infty}A_n R^{-n}\int_0^{2\pi}\sin n\phi\sin m\phi\, d\phi = A_m R^{-m}$$

$$\int_0^{2\pi}\sin n\phi\sin m\phi\, d\phi=\pi\delta_{mn}$$

$$=\frac{2V_0}{m\pi}\left\{-\cos m\phi\Big|_0^{\pi/2}+\cos m\phi\Big|_{\pi/2}^{\pi}\right\}=\frac{2V_0}{m\pi}\left\{-2\cos m\pi/2+1+\cos m\pi\right\}$$

$$\left\{-2\cos m\frac{\pi}{2}+1+\cos m\pi\right\}=0\quad (m=1)$$
$$=4\quad (m=2)$$
$$=0\quad (m=3,4,5)$$
$$=4\quad (m=6)$$

$$\frac{1}{\pi}\int_0^{2\pi}V(R,\phi)\sin m\phi\, d\phi=\frac{8V_0}{m\pi}=A_m R^{-m}$$

$$\rightarrow \underline{A_m=\frac{8V_0R^m}{m\pi};\quad m=4n-2,\quad n=1,2,3,\ldots}$$

$$\underline{V(r,\phi)=\frac{8V_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n-2}\frac{R^{4n-2}}{r^{4n-2}}\sin[(4n-2)\phi]}$$

5

Solve Laplace's equation for a single line charge (radius a), charge density λ .
You may need to use Gauss's Law to determine one of the unknown constants.

$$V(r,\phi) = A\ln r + B; \quad \mathbf{E} = -\nabla V = -\frac{A}{r}\hat{r}$$

Gauss's Law

$$2\pi r L E_r = \frac{\lambda L}{\varepsilon_0}; \quad E_r = \frac{\lambda}{2\pi\varepsilon_0 r} \quad \rightarrow \quad \underline{A = -\frac{\lambda}{2\pi\varepsilon_0}}$$

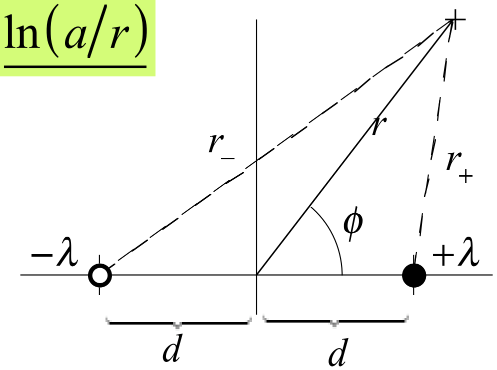
Potential on the surface of line charge

$$V(a,\phi) = 0 = -\frac{\lambda}{2\pi\varepsilon_0}\ln a + B; \quad B = \frac{\lambda}{2\pi\varepsilon_0}\ln a$$

$$V(r,\phi) = \left(-\lambda/2\pi\varepsilon_0\right)\ln r + \left(\lambda/2\pi\varepsilon_0\right)\ln a = \left(\lambda/2\pi\varepsilon_0\right)\underline{\ln\left(a/r\right)}$$

Find potential of Line Dipole:

$$V_d(r,\phi) = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{a}{r_+} + \frac{-\lambda}{2\pi\varepsilon_0}\ln\frac{a}{r_-} = \frac{\lambda}{2\pi\varepsilon_0}\left(\ln r_- - \ln r_+\right)$$



$$r_\pm = r\left(1 \mp 2d\cos\phi/r + d^2/r^2\right)^{1/2}$$

$$\ln r_+ = \ln r + \frac{1}{2}\ln\left(1 - 2d\cos\phi/r + d^2/r^2\right)$$

$$\ln r_- = \ln r + \frac{1}{2}\ln\left(1 + 2d\cos\phi/r + d^2/r^2\right)$$

$$V_d(r,\phi) = \frac{\lambda}{2\pi\varepsilon_0}\left(\ln r_- - \ln r_+\right)$$

$$\approx \frac{2\lambda d\cos\phi}{2\pi\varepsilon_0 r} = \underline{\frac{p_{line}\cos\phi}{2\pi\varepsilon_0 r}, \text{ where } p_{line} = 2\lambda d}$$