PHY481: Electromagnetism

Magnetic fields (reprise) & vector potential

Magnetic force and field

Magnetic force: $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$

Lorentz force: $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

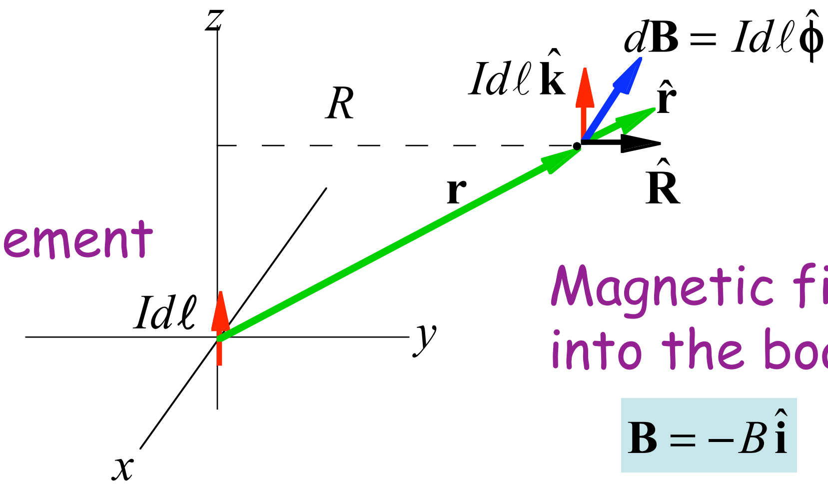
Right hand rule #1

Currents make magnetic fields: Biot-Savart Law

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{Id\boldsymbol{\ell} \times \hat{\mathbf{r}}}{r^2}$$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_{wire} \frac{Id\boldsymbol{\ell} \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d^3x'$$

Current element

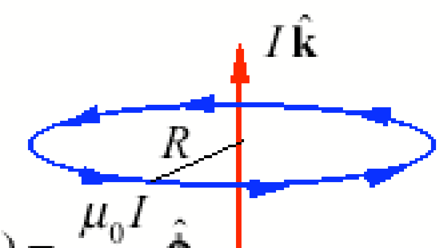


Magnetic field into the board

$\mathbf{B} = -B\,\hat{\mathbf{i}}$

Applications:

Short and long wires carrying current
End of a wire carrying current
Ring of current on axis



$$\mathbf{B}(R) = \frac{\mu_0 I}{2\pi R}\hat{\boldsymbol{\phi}}$$

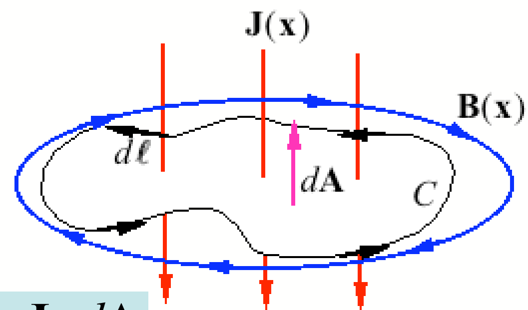
Ampere's Law

$\nabla \cdot \mathbf{B} = 0$ Magnetic fields are "solenoidal" (always true)

$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ Ampere's Law (constant currents only.
Maxwell's equations have another term`

$\oint_C \mathbf{B} \cdot d\ell = \mu_0 I_{encl}$ Via Stokes's theorem:

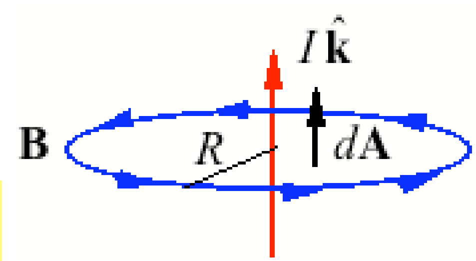
$\oint_C \mathbf{B} \cdot d\ell = \mu_0 \oint_S \mathbf{J} \cdot d\mathbf{A}$



$dI = \mathbf{J} \cdot d\mathbf{A}$

Ampere's Law and symmetries

Straight wire carrying current
Field by the easy way
Use right hand rule #2

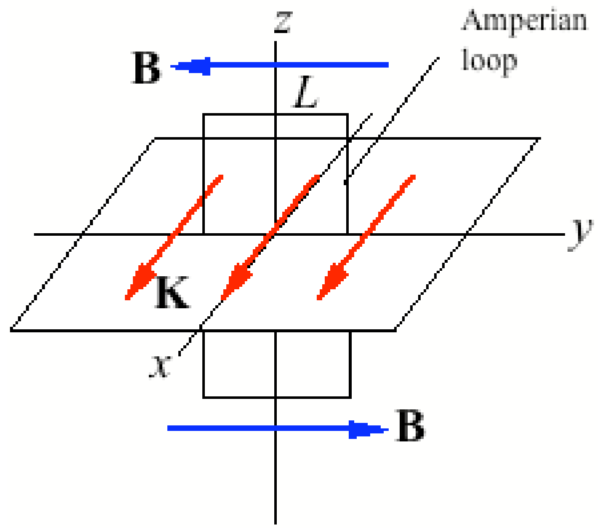


$\mathbf{B}(R) = \frac{\mu_0 I}{2\pi R}\hat{\boldsymbol{\phi}}$

More Ampere's law applications

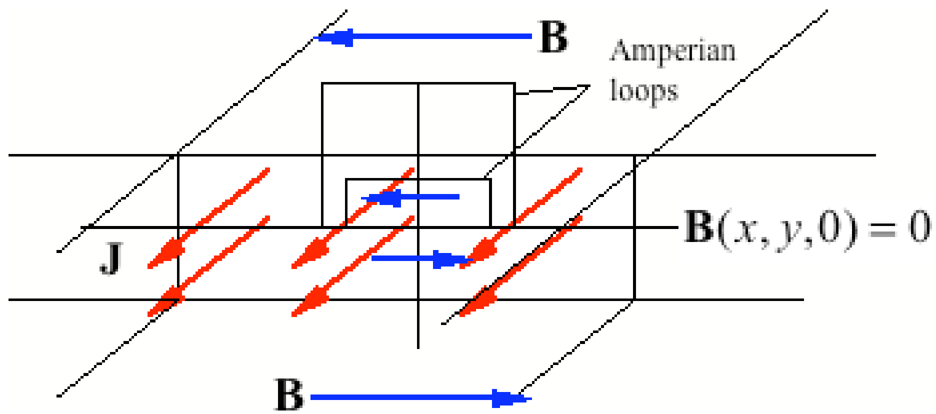
Infinite current sheet

$$\mathbf{B} = -\frac{\mu_0 K}{2}\hat{\mathbf{j}} \quad z > 0; \quad \mathbf{B} = +\frac{\mu_0 K}{2}\hat{\mathbf{j}} \quad z < 0$$



Infinite current slab

Inside	Outside
$\mathbf{B}_{top} = -\mu_0 Jz\,\hat{\mathbf{j}}$	$\mathbf{B}_{top} = -\mu_0 Ja\,\hat{\mathbf{j}}$
$\mathbf{B}_{bottom} = \mu_0 Jz\,\hat{\mathbf{j}}$	$\mathbf{B}_{bottom} = \mu_0 Ja\,\hat{\mathbf{j}}$



Field equations for B(x) & Vector potential A(x)

Always true | Constant currents only.

Constant currents:

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \cdot (\nabla \times \mathbf{B}) = 0 = \mu_0 \nabla \cdot \mathbf{J}$$

$$\nabla \cdot \mathbf{J} = 0$$

Ampere's Law:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \qquad \oint_C \mathbf{B} \cdot d\ell = \mu_0 I_{encl}$$

Get field from a Vector Potential A

$$\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x})$$

Fields from potentials

$$\mathbf{E} = -\nabla V \text{ now also } \mathbf{B} = \nabla \times \mathbf{A}$$

Start with Biot-Savart Law:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d^3x'$$

Previously used Identity

$$\frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} = -\nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|}$$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{x}') \times \left[-\nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right] d^3x'$$

Show

$$\mathbf{J}(\mathbf{x}') \times \left(-\nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = \nabla \times \left(\qquad \right)$$

Required Identity

Prove: $\mathbf{J}(\mathbf{x}') \times \left(-\nabla \frac{1}{|\mathbf{x}-\mathbf{x}'|} \right) = \nabla \times \left(\qquad \right)$

Let $f(\mathbf{x}) = \frac{1}{|\mathbf{x}-\mathbf{x}'|}$

$$\begin{aligned}
\mathbf{J}(\mathbf{x}') \times \left[-\nabla f(\mathbf{x}) \right] &= \varepsilon_{ijk} J_j(\mathbf{x}') \left[-\nabla f(\mathbf{x}) \right]_k \\
&= -\varepsilon_{ijk} J_j(\mathbf{x}') \frac{\partial f(\mathbf{x})}{\partial x_k} = \varepsilon_{ikj} \frac{\partial}{\partial x_k} \left[f(\mathbf{x}) J_j(\mathbf{x}') \right] \\
&= \nabla \times \left[f(\mathbf{x})\, \mathbf{J}(\mathbf{x}') \right]
\end{aligned}$$

$$\left[\nabla f(\mathbf{x}) \right]_k = \frac{\partial f(\mathbf{x})}{\partial x_k}$$

$$\varepsilon_{ijk} = -\varepsilon_{ikj}$$

$$\mathbf{J}(\mathbf{x}') \times \left(-\nabla \frac{1}{|\mathbf{x}-\mathbf{x}'|} \right) = \nabla \times \left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \right)$$

Field equations for B(x) & Vector potential A(x)

Biot-Savart Law: $$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} d^3x'$$

$$\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} = -\nabla \frac{1}{|\mathbf{x}-\mathbf{x}'|}$$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{x}') \times \left[-\nabla \frac{1}{|\mathbf{x}-\mathbf{x}'|}\right] d^3x'$$

$$\left(\nabla \frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \times \mathbf{J}(\mathbf{x}') = \nabla \times \left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right)$$

$$\mathbf{B}(\mathbf{x}) = \nabla \times \left[\frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}') d^3x'}{|\mathbf{x}-\mathbf{x}'|}\right]$$

$$\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x})$$

Fields from potentials

Vector potential: $$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}') d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

$$\mathbf{E} = -\nabla V$$
$$\mathbf{B} = \nabla \times \mathbf{A}$$

Vector potential *A* plays a crucial role in the quantization of the EM field

Vector potential calculations

If currents are bounded (do not extend to infinity) get A by

Volume currents
$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

Vector potential sums vectors that point in the direction of the current!

Surface currents
$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{x}')d^2x'}{|\mathbf{x}-\mathbf{x}'|}$$

Line currents
$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{Id\ell}{|\mathbf{x}-\mathbf{x}'|}$$

$\mathbf{I} = Id\ell$ is a vector!

"Real" currents form loops and thus are bounded

If currents extend to infinity
find B, then get A via Stokes's theorem:

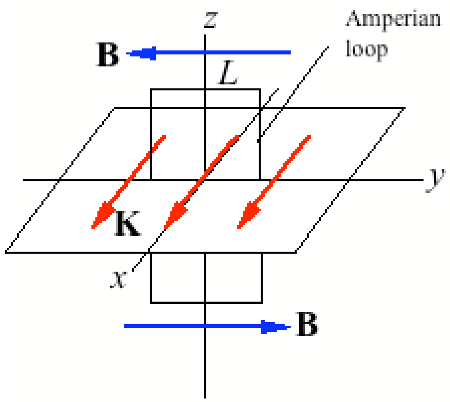
$$\oint_C \mathbf{A}\cdot d\ell = \int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \int \mathbf{B}\cdot d\mathbf{a}$$

For example, a long straight wire !

Infinite current sheet

Ampere's law around the loop shown

Counter clockwise around the loop



Magnetic field B is easy:

$$\oint \mathbf{B} \cdot d\ell = \mu_0 I_{encl}$$

$$\mathbf{B} = \frac{\mu_0 K}{2} \begin{cases} -\hat{\mathbf{j}} & z > 0 \\ +\hat{\mathbf{j}} & z < 0 \end{cases}$$

Vector potential **A**:

For bounded currents

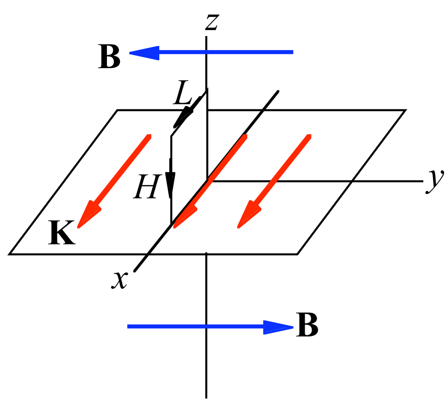
$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}')d^3x'}{|\mathbf{x} - \mathbf{x}'|}$$

Currents of infinite extent

$$\oint_C \mathbf{A} \cdot d\ell = \int \mathbf{B} \cdot d\mathbf{a}$$

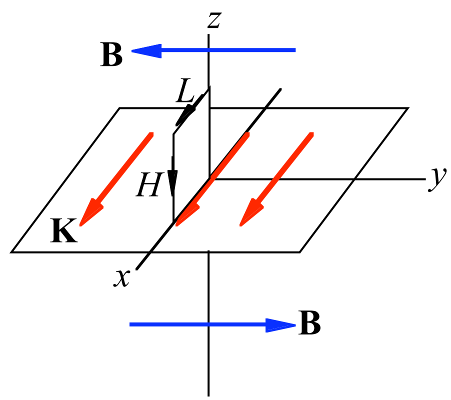
But infinite extent currents form loops

Choose A•dl loop with normal in one B direction



Aside

Magnetic field **B** is only in y direction



$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$B_2 = \varepsilon_{231}\frac{\partial A_1}{\partial x_3} - \varepsilon_{231}\frac{\partial A_3}{\partial x_1}$$

Involves *z* dependence of A_x and, x dependence of A_z.

Assume $\mathbf{A}(x,z) = f(z)\,\hat{\mathbf{i}} + g(x)\,\hat{\mathbf{k}}$ $\quad f(0)=0$ and $g(0)=0$

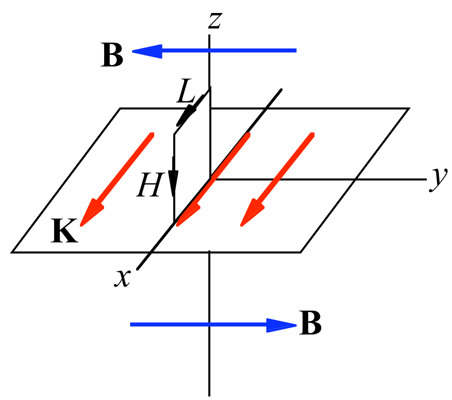
Infinite current sheet

Vector potential **A**:

For currents of infinite extent use

$$\oint_C \mathbf{A}\cdot d\ell = \int \mathbf{B}\cdot d\mathbf{a}$$

Choose A•dl loop with normal in one B direction



$$\int \mathbf{B}\cdot d\mathbf{a} = -BHL$$

Assume

$$\mathbf{A}(x,z) = f(z)\,\hat{\mathbf{i}} + g(x)\,\hat{\mathbf{k}}$$

$$\oint_C \mathbf{A}\cdot d\ell = f(H)\int_0^L dx + g(L)\int_H^0 dz = Lf(H) - Hg(L)$$

Find functions *f* and *g* such that: $Lf(H) - Hg(L) = -BHL$

$z > 0$

$$f(z) = -Bz/2;\, g(x) = Bx/2$$

$$\mathbf{A}(x,z) = \frac{B}{2}\left(-z\,\hat{\mathbf{i}} + x\,\hat{\mathbf{k}}\right)$$

$z < 0$

$$f(z) = +Bz/2;\, g(x) = -Bx/2$$

$$\mathbf{A}(x,z) = \frac{B}{2}\left(+z\,\hat{\mathbf{i}} - x\,\hat{\mathbf{k}}\right)$$

Yes! $\mathbf{B} = \nabla\times\mathbf{A}$

Also in Coulomb gauge: $\nabla\cdot\mathbf{A} = 0$

Visualizing the vector potential

Infinite sheet of current

Magnetic field

$$\mathbf{B} = \frac{\mu_0 K}{2}\begin{cases} -\hat{\mathbf{j}} & z > 0 \\ +\hat{\mathbf{j}} & z < 0 \end{cases}$$

Changes sign above and below the sheet

$$B/2 = \mu_0 K/4$$

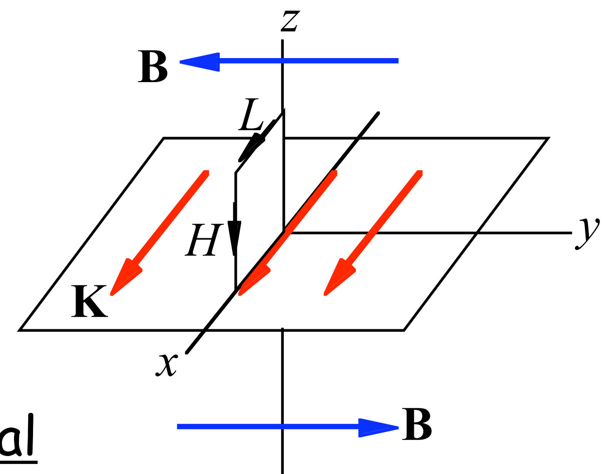
Vector potential

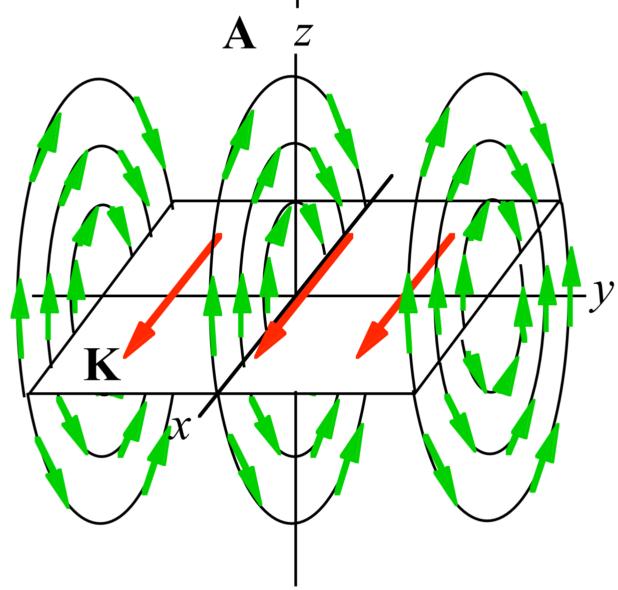
$$\mathbf{A} = \mu_0 K\left(-z\,\hat{\mathbf{i}} + x\,\hat{\mathbf{k}}\right)/4 \quad z > 0$$

$$\mathbf{A} = \mu_0 K\left(+z\,\hat{\mathbf{i}} - x\,\hat{\mathbf{k}}\right)/4 \quad z < 0$$

A_z is discontinuous across the current sheet.

Why does **A** point opposite to **K** and curve this way?





Solenoid field and vector potential

Magnetic field

Ampere's law gives

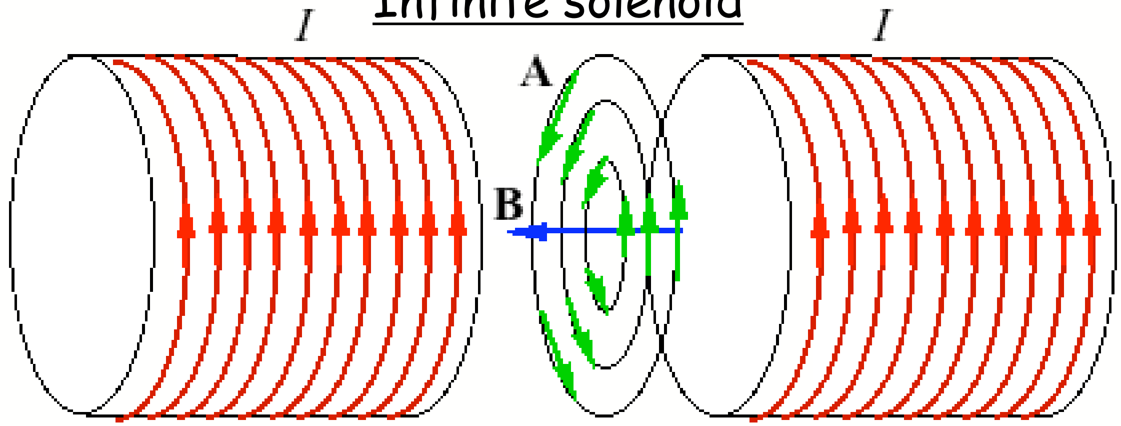
$$\mathbf{B} = -B\,\hat{\mathbf{j}} = -\mu_0 n I\,\hat{\mathbf{j}}$$

$n = \text{turns/unit length}$

Vector potential

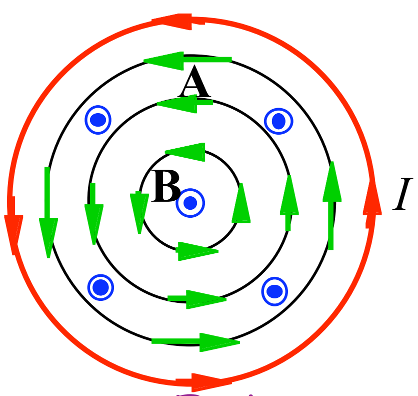
$$\mathbf{A} = B\left(-z\,\hat{\mathbf{i}} + x\,\hat{\mathbf{k}}\right)/2$$

Infinite solenoid

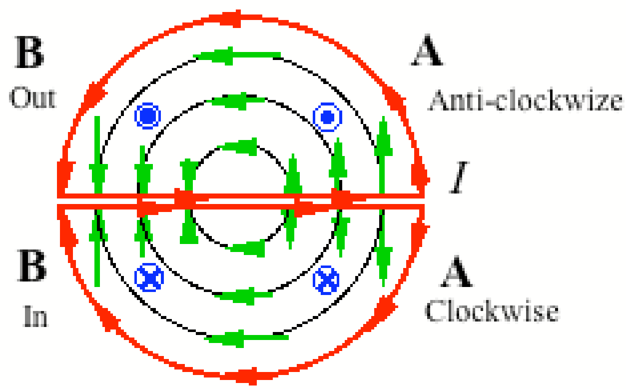


(uniform), and *growing* circular vector potential ***A***.

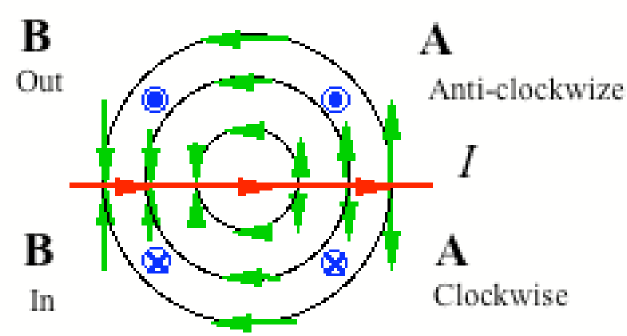
Full solenoid



Two half solenoids



Current sheet



End views. N... ...pare with a current sheet.