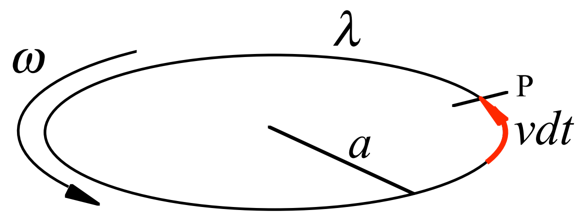
PHY481: Electromagnetism

Homework 7
Magnetic quadrupole doublet

HW 7

Problem 7.2

$$dQ = \lambda v dt$$
$$I = \frac{dQ}{dt} = \lambda v$$

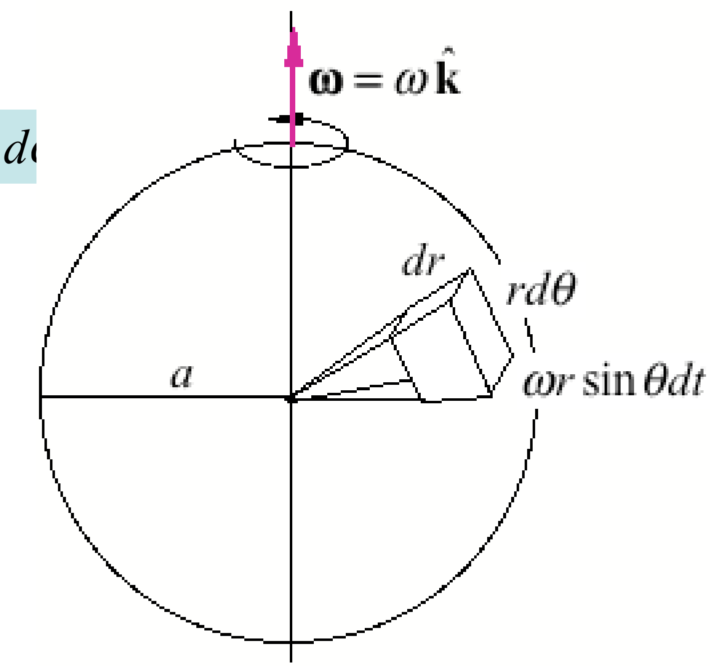


$$I = \lambda v = \lambda \omega a$$

Problem 7.3

Quick : $\mathbf{J}(r) = qn\mathbf{v} = \rho_c \boldsymbol{\omega} \times \mathbf{r} = \rho_c \omega r \sin\theta \hat{\boldsymbol{\phi}}$

Slow but sure : $dA = rdrd\theta$ $d^3x = r^2 dr \sin\theta d\theta d\phi$ $\omega = d$



$$\int_S \mathbf{J}(r) \cdot d\mathbf{A} = \frac{d}{dt} \int_V \rho_c d^3x$$

$$\int_S J_\phi(r) rdrd\theta = \int_V \left(\rho_c r \sin\theta \frac{d\phi}{dt} \right) rdrd\theta$$
$$J_\phi(r) = \rho_c r \sin\theta \omega$$

$$\mathbf{J}(r) = \rho_c r \sin\theta \omega \hat{\boldsymbol{\phi}}$$

HW 7 (cont'd)

Problem 7.5

$$A = \frac{\pi}{4}D^2; \quad A' = \frac{\pi}{4}(D')^2 = \frac{\pi}{4}(D/2)^2 = \frac{A}{4}$$

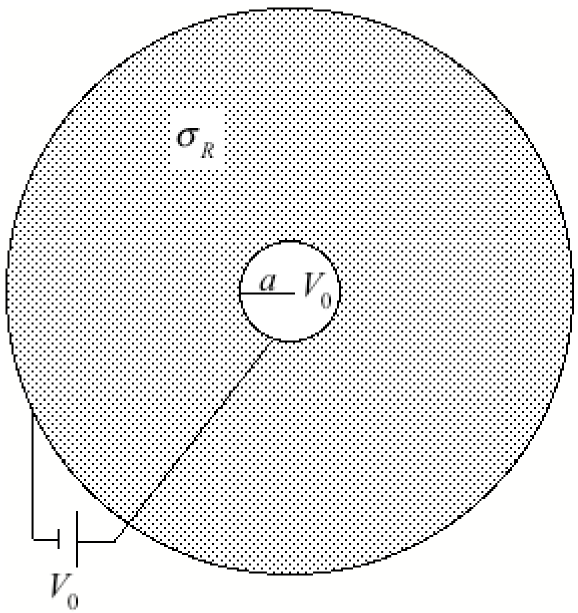
$$R' = \frac{\rho_R L'}{A'} = \frac{\rho_R 2L}{A/4} = 8\frac{\rho_R L}{A} = 8R$$

Problem 7.7

$$V(r) = \frac{Q_0}{4\pi\varepsilon_0 r}; \qquad \frac{Q_0}{4\pi\varepsilon_0 a} = V_0 \qquad Q_0 = V_0 4\pi\varepsilon_0 a$$

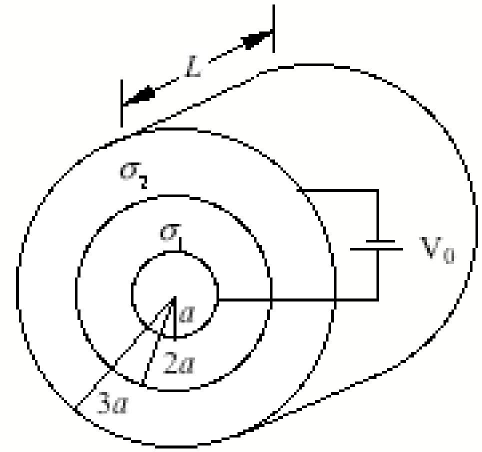
$$= \frac{V_0 a}{r}$$

$$\mathbf{E} = -\nabla V = \frac{V_0 a}{r^2}\hat{\mathbf{r}} \qquad \mathbf{J}(r) = \sigma_R \mathbf{E} = \frac{\sigma_R V_0 a}{r^2}\hat{\mathbf{r}}$$



HW 7 (cont'd)

7.9 Cylinders length L , resistive media between radius a and $3a$. What is the resistance? What is the charge density on the interface at $2a$?



General solutions

$$V_1(r) = A\ln r + B$$
$$V_2(r) = C\ln r + D$$

Current densities

$$J_1(r) = \sigma_1 E_1 = \sigma_1\left(-\nabla V_1\right) = -\sigma_1 A/r$$
$$J_2(r) = \sigma_2 E_2 = \sigma_2\left(-\nabla V_2\right) = -\sigma_2 C/r$$

Boundary Conditions

$$V_1(a) = V_0$$
$$V_2(3a) = 0$$
$$V_1(2a) = V_2(2a)$$
$$J_1(2a) = J_2(2a)$$

Determine coefficients

$$B = V_0 - A\ln a; \quad C = A\sigma_1/\sigma_2$$
$$D = -C\ln 3a = -A\sigma_1/\sigma_2 \ln 3a$$

At $r = 2a$: $A\ln(2a) + V_0 - A\ln(a) = A\ln(2a)\sigma_1/\sigma_2 - A\sigma_1/\sigma_2\ln 3a$

$$A\left[(\sigma_1/\sigma_2)\ln\left(\tfrac{2}{3}\right) + \ln\left(\tfrac{1}{2}\right)\right] = V_0 \rightarrow \underline{A = V_0/\left[(\sigma_1/\sigma_2)\ln\left(\tfrac{2}{3}\right) + \ln\left(\tfrac{1}{2}\right)\right]}$$

$$\sigma_{free} = \varepsilon_0 \Delta E = \varepsilon_0 (A - C)/2a = \frac{\varepsilon_0 V_0(\sigma_1/\sigma_2 - 1)}{2a\left[\ln\left(\tfrac{3}{2}\right) + \ln(2)\sigma_1/\sigma_2\right]}$$

$$I = \int_{r=a} J dA = J_1(a)[2\pi a L] = \frac{2\pi L\sigma_1 V_0}{\left[(\sigma_1/\sigma_2)\ln\left(\tfrac{3}{2}\right) + \ln(2)\right]}$$

$$R = \frac{V_0}{I} = \frac{\left[(\sigma_1/\sigma_2)\ln\left(\tfrac{3}{2}\right) + \ln(2)\right]}{2\pi L\sigma_1}$$

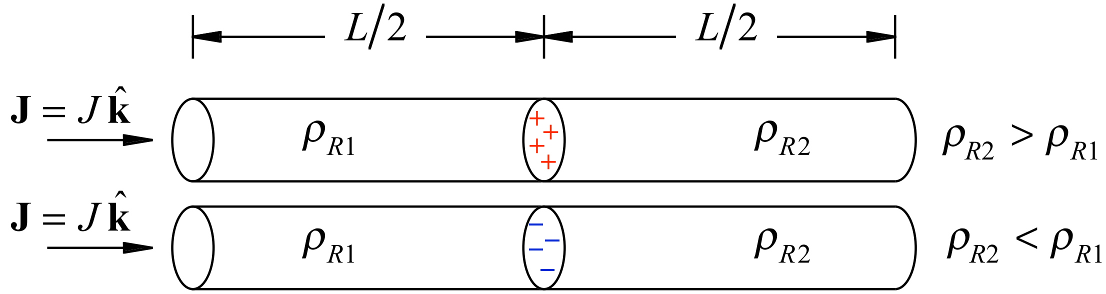
HW 7 (cont'd)

Problem 7.11

$$E_1 = J\rho_{R1}; \quad E_2 = J\rho_{R2}$$

$$\sigma_c^{free} = \varepsilon_0\left(E_2 - E_1\right) = \varepsilon_0 J\left(\rho_{R2} - \rho_{R1}\right)$$

$$\sigma_c^{free} = \varepsilon_0 \frac{I\left(\rho_{R2} - \rho_{R1}\right)}{A}$$



Problem 7.15

$$Q(t) = Q_0 e^{-t/\tau} = Q_0 e^{-t/RC}$$

$$Q(\tau)/Q_0 = Q(2\tau)/Q(\tau) = Q(6\tau)/Q(5\tau) = e^{-1}$$

$$f(t) = Q(t)/Q_0 = e^{-t/\tau}; \quad t = -\tau \ln f$$

$$t_{f=0.1} = -\tau \ln(0.1) = 2.3\tau$$

Change in stored energy in time *t*:

$$\Delta\varepsilon(t) = \frac{1}{2}\left(QV - Q_0 V_0\right) = \frac{Q^2(t) - Q_0^2}{2C} = \frac{Q_0^2\left(e^{-2t/RC} - 1\right)}{2C}$$

zero sum

Energy dissipation from resistor in time t:

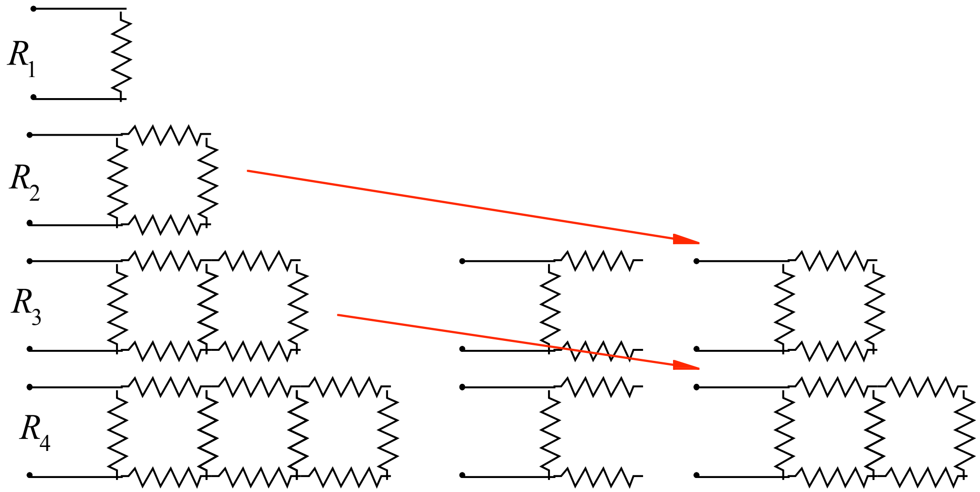
$$I(t) = \frac{dQ(t)}{dt} = -\frac{Q(t)}{RC}$$

$$\varepsilon_R(t) = \int_0^t dt P(t) = \int_0^t dt (I^2 R) = \frac{1}{RC^2}\int_0^t dt Q^2(t) = \frac{Q_0^2}{2C}\left[1 - e^{-2t/RC}\right]$$

HW 7 (cont'd)

Problem 7.18

$$R_1 = R$$
$$1/R_2 = 1/R + 1/(2R + R_1)$$
$$1/R_3 = 1/R + 1/(2R + R_2)$$
$$1/R_n = 1/R + 1/(2R + R_{n-1})$$

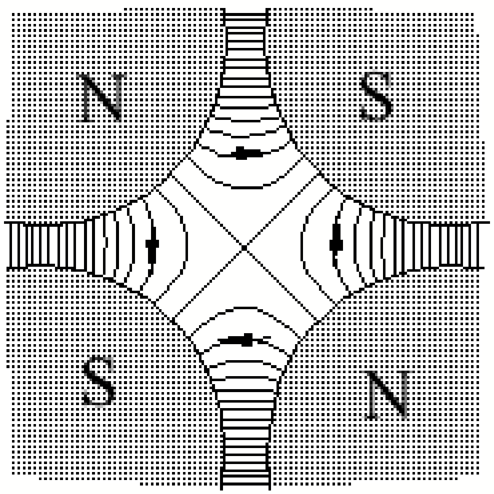


$$R_n = \frac{R(2R + R_{n-1})}{3R + R_{n-1}}$$
$$R_\infty = \frac{R(2R + R_\infty)}{3R + R_\infty}$$

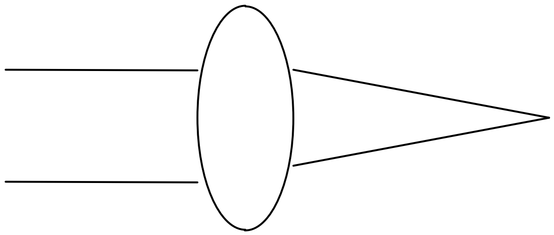
$$R_\infty^2 + 2RR_\infty - 2R^2 = 0$$
$$R_\infty = R(\sqrt{3} - 1)$$

Quadrupole

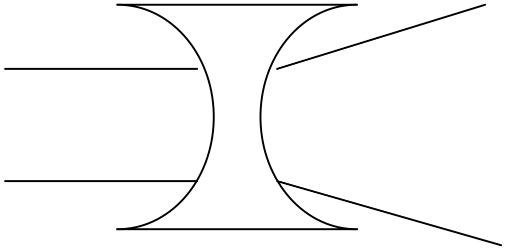
Protons coming at you



left or right, focused

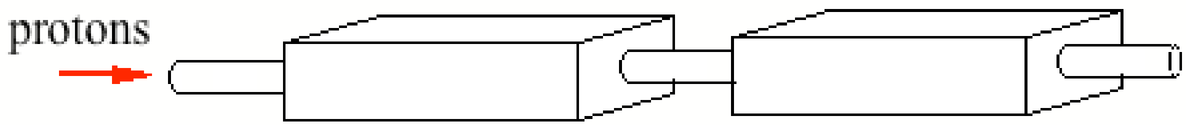


top or bottom, defocused



The quadrupole doublet

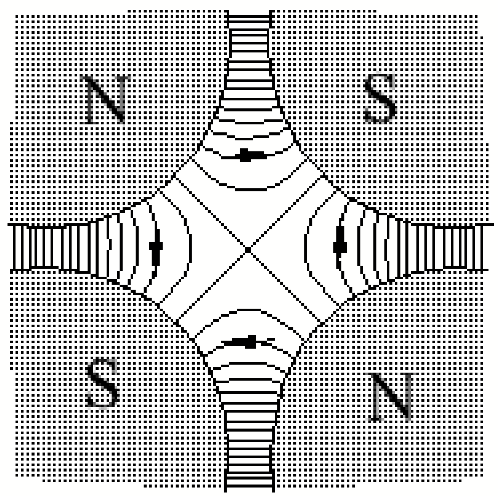
The quadrupole doublet. Focuses (weakly) in both dimensions

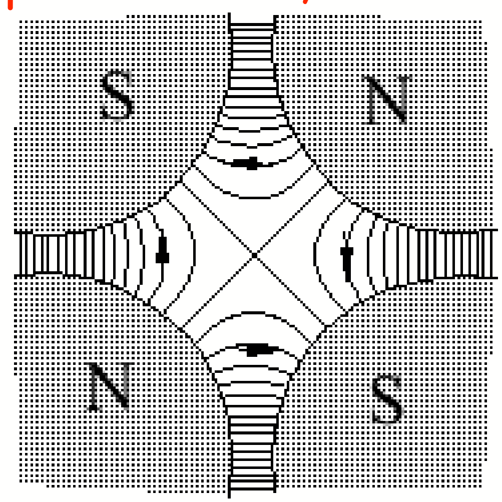


left or right, focused
top or bottom, defocused

left or right, defocused
top or bottom, focused

Protons
coming
at you





left or right, closer inward (field small)

top or bottom, further out (field large)

Vector potential calculations

If currents are bounded (do not extend to infinity) get A by

Volume currents
$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

Surface currents
$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{x}')d^2x'}{|\mathbf{x}-\mathbf{x}'|}$$

Line currents
$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{I d\ell}{|\mathbf{x}-\mathbf{x}'|}$$

If currents extend to infinity get A via Stokes's theorem and B:

$$\oint_C \mathbf{A}\cdot d\ell = \int (\nabla \times \mathbf{A})\cdot d\mathbf{a} = \int \mathbf{B}\cdot d\mathbf{a}$$