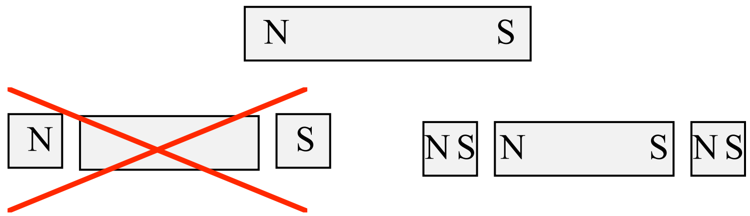
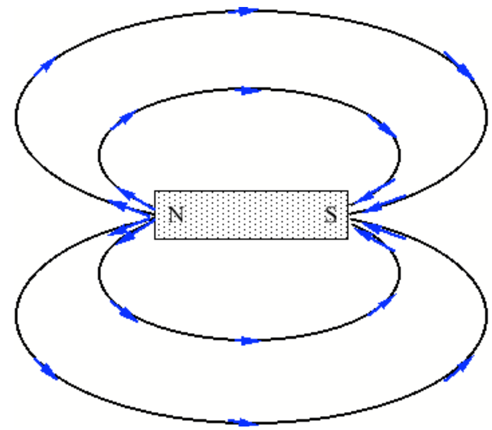
PHY481: Electromagnetism

Magnetic dipole

Magnetic dipole

In the many attempts to find a magnetic monopole, no one as ever seen one.





$$\nabla \cdot \mathbf{B} = 0 \;\;\Rightarrow\;\; \text{no magnetic monopole}$$

Maybe there is only one in the universe (Dirac and monopoles)

The vector potential of an arbitrary current distribution can tough to calculate.
However it can be described in a "multipole" expansion.

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

Legendre polynomials

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{\sqrt{r^2 - 2rr'\cos\theta + r'^2}} = \frac{1}{r} + \frac{r'\cos\theta}{r^2} + \frac{(r')^2}{r^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \ldots$$

$$P_1(\cos\theta) = \cos\theta$$

$$P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}$$

Also used for electric "multipole" expansion earlier in semester

Magnetic multi-pole expansion

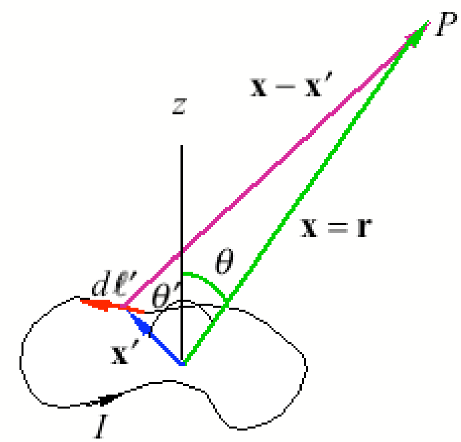
Vector potential of a thin current loop

vectors

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|} \qquad \mathbf{J}(\mathbf{x}')d^3x' = I\,d\boldsymbol{\ell}'$$

$$= \frac{\mu_0 I}{4\pi r}\left[\int d\boldsymbol{\ell}' + \frac{1}{r}\int r' P_1(\cos\theta')\,d\boldsymbol{\ell}' + \frac{1}{r^2}\int (r')^2 P_2(\cos\theta')\,d\boldsymbol{\ell}' + \ldots\right.$$

monopole=0 dipole quadrupole



If this loop lies in x-y plane

ψ' is angle between $\mathbf{x}'$ and $d\boldsymbol{\ell}'$; $|\mathbf{x}'| = r'$

$\hat{\mathbf{n}} = \hat{\mathbf{k}}$ is normal to the plane

scalars

dipole integral

$$I\int r'\cos\theta'\,d\boldsymbol{\ell}' = \left[\frac{I}{2}\int r'\sin\psi'\,d\ell'\right]\hat{\mathbf{n}}\times\hat{\mathbf{r}}$$

(mathematical wizardry)
(P&S 8.71 - 8.77)

Dipole moment

$$m = \left[\frac{I}{2}\int r'\sin\psi'\,d\ell'\right] \qquad \mathbf{m} = m\hat{\mathbf{n}}$$

Not dependent on *P*

Far from the loop

$\hat{\mathbf{n}} = \hat{\mathbf{k}}$

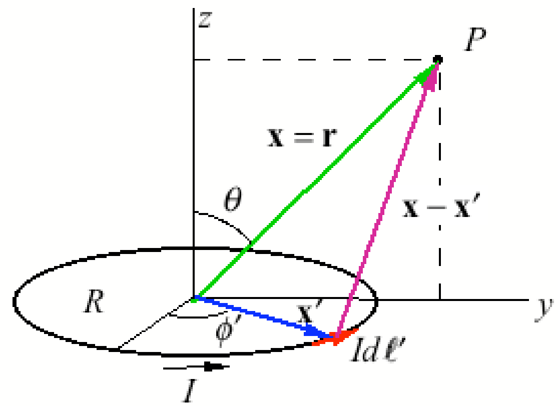
$$\mathbf{A}(r,\theta) \approx \frac{\mu_0 I}{4\pi r^2}\int r'\cos\theta'\,d\boldsymbol{\ell}' = \frac{\mu_0 m\sin\theta}{4\pi r^2}\hat{\boldsymbol{\phi}}$$

$$\mathbf{B}(r,\theta) = \nabla\times\mathbf{A}(r,\theta) = \frac{\mu_0 m}{4\pi r^3}\left[2\hat{\mathbf{r}}\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right]$$

Another approach

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 I}{4\pi}\int \frac{I\,d\ell'}{|\mathbf{x}-\mathbf{x}'|}$$

$$d\ell' = R\,d\phi'\left(-\hat{\mathbf{i}}\sin\phi' + \hat{\mathbf{j}}\cos\phi'\right)$$



$$\mathbf{x} = \hat{\mathbf{j}}\,y + \hat{\mathbf{k}}z$$
$$\mathbf{x}' = \hat{\mathbf{i}}\,R\cos\phi' + \hat{\mathbf{j}}\,R\sin\phi'$$
$$\mathbf{x}-\mathbf{x}' = -\hat{\mathbf{i}}\,R\cos\phi' + \hat{\mathbf{j}}(y - R\sin\phi') + \hat{\mathbf{k}}z$$

$$|\mathbf{x}-\mathbf{x}'|^2 = R^2\cos^2\phi' + (R\sin^2\phi' + y^2 - 2yR\sin\phi') + z^2$$
$$|\mathbf{x}-\mathbf{x}'| = \sqrt{R^2 + y^2 + z^2 - 2yR\sin\phi'}$$

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 IR}{4\pi}\int_0^{2\pi} d\phi' \frac{\left(-\hat{\mathbf{i}}\sin\phi' + \hat{\mathbf{j}}\cos\phi'\right)}{\sqrt{R^2 + y^2 + z^2 - 2yR\sin\phi'}} = \frac{\mu_0 IR}{4\pi}\int_0^{2\pi} d\phi' \frac{\left(-\hat{\mathbf{i}}\sin\phi' + \hat{\mathbf{j}}\cos\phi'\right)}{\sqrt{R^2 + r^2 - 2rR\sin\theta\sin\phi'}}$$

$A_y\,\hat{\mathbf{j}}$ @ $\phi' = 0, \pi$ have equal distance to P, average to zero

if point P is placed elsewhere $-\hat{\mathbf{i}}$ direction becomes $\hat{\boldsymbol{\phi}}$

$$A_\phi = \frac{\mu_0 IR}{4\pi}\int_0^{2\pi} \frac{\sin\phi'\,d\phi'}{\sqrt{R^2 + r^2 - 2rR\sin\theta\sin\phi'}}$$

Now introduce Legendre Polynomials

Compare with expansion on slide 2

$$\frac{1}{\sqrt{r^2+R^2-2rR\cos\theta}}=\frac{1}{r}+\frac{RP_1(\cos\theta)}{r^2}+\ldots$$

$$\cos\theta\rightarrow\sin\theta\sin\phi'$$

$$A_\phi=\frac{\mu_0 IR}{4\pi r}\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^n\int_0^{2\pi}P_n(\sin\theta\sin\phi')\sin\phi' d\phi'$$

A = 0, for all even n, with first non zero term, n = 1

$$\int_0^{2\pi}(\sin^2 x)dx=\pi$$

$$A_\phi\approx\frac{\mu_0 IR^2\sin\theta}{4\pi r^2}\int_0^{2\pi}\sin^2\phi' d\phi'=\frac{\mu_0 I\pi R^2\sin\theta}{4\pi r^2}$$

$$A_\phi\approx\frac{\mu_0 m\sin\theta}{4\pi r^2};\quad m=I\pi R^2$$

$$B_\theta=-\frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right)\approx\frac{\mu_0 m}{4\pi}\left(-\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\sin\theta}{r}\right)\right)=\frac{\mu_0 m}{4\pi r^3}\sin\theta$$

$$B_r=\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right)\approx\frac{\mu_0 m}{4\pi r^3\sin\theta}\frac{\partial}{\partial\theta}\left(\sin^2\theta\right)=\frac{\mu_0 m}{4\pi r^3}(2\cos\theta)$$

$$\mathbf{B}\approx\frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}})$$

point dipole

$$\mathbf{B}_{\text{point dipole}}=\frac{\mu_0}{4\pi}\left[\frac{3\hat{\mathbf{r}}\left(\mathbf{m}\cdot\hat{\mathbf{r}}\right)-\mathbf{m}}{r^3}\right];\quad\mathbf{m}=m\hat{\mathbf{k}}$$

General expressions for magnetic dipole

Dipole moment

$$\mathbf{m} = \frac{1}{2}\int \mathbf{x}' \times \mathbf{J}(\mathbf{x}')d^3x'$$

Vector potential

$$\hat{\mathbf{m}} \times \hat{\mathbf{r}} = \int \mathbf{J}(\mathbf{x}')\left[\mathbf{x}' \cdot \hat{\mathbf{r}}\right]d^3x'$$

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\hat{\mathbf{m}} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\frac{\hat{\mathbf{m}} \times \mathbf{x}}{|\mathbf{x}|^3}$$

Magnetic field

$$\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{r}}(\mathbf{m} \cdot \hat{\mathbf{r}}) - \mathbf{m}}{r^3}$$

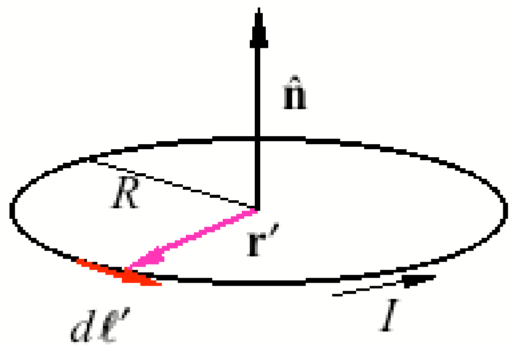
Torque on a magnetic dipole

$$\mathbf{N} = \mathbf{m} \times \mathbf{B}$$

Magnetic dipole potential energy

$$U = -\mathbf{m} \cdot \mathbf{B}$$

Circular loop



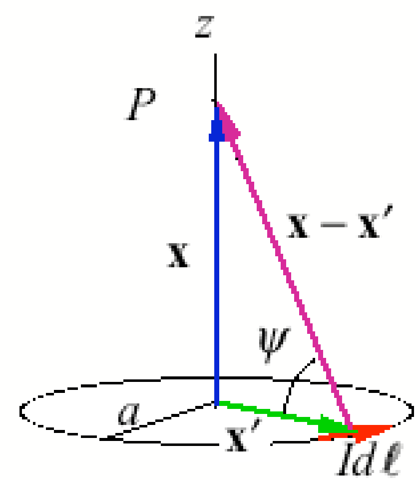
$$\mathbf{m} = \frac{IR}{2}\int \hat{\mathbf{r}} \times d\ell = \frac{IR}{2}\int_0^{2\pi} R\,d\phi\,\hat{\mathbf{n}}$$

$$\mathbf{m} = I\pi R^2\hat{\mathbf{n}} = IA\hat{\mathbf{n}}$$

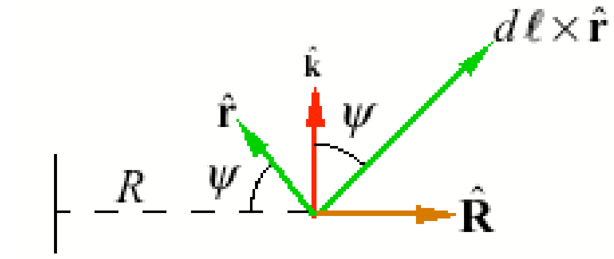
HW8 hints

Problem 8.10

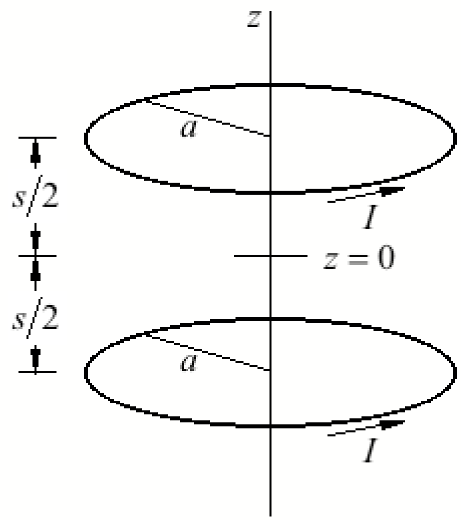
$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x})\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi}\int \frac{d\ell\times\hat{\mathbf{r}}}{r^2}$$



$$d\ell = r d\phi\,\hat{\boldsymbol{\phi}}$$



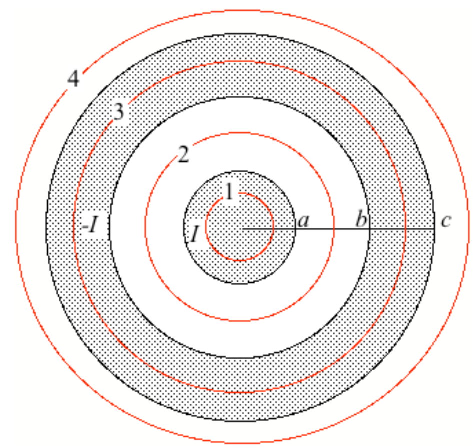
$$\frac{d^2 B_z(0)}{dz^2} = 0$$



HW8 Hints

Problem 8.14 Constraints: $\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$

Problem 8.18

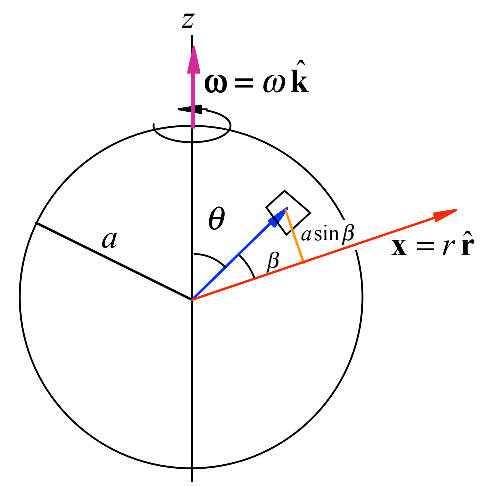


Ampere's Law: $\int \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{encl} = \mu_0 \int \mathbf{J} \cdot d\mathbf{A}$

Problem 8.22 $\mathbf{A}_{in}$ & $\mathbf{A}_{out}$ given in text

$$\mathbf{B}_{in} = \nabla \times \mathbf{A}_{in} = \nabla \times \frac{\mu_0 \sigma a \omega}{3} (\hat{\mathbf{k}} \times r\hat{\mathbf{r}})$$

$$\mathbf{B}_{out} = \nabla \times \mathbf{A}_{out} = \nabla \times \left(\frac{\mu_0 \sigma a^4 \omega}{3} \right) \left(\hat{\mathbf{k}} \times \frac{\hat{\mathbf{r}}}{r^2} \right)$$



Problem 8.34 Use result of Problem 8.10