PHYSICS 321 - Classical Mechanics I

Two Practice Exams! **Practice Exam #1**

Total points = 30. There are only 3 problems on this exam. Think about what is going on in each problem, and show all your work. Include units in your answers.

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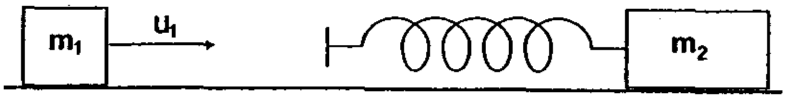
1. [11] A block of mass m_1=0.4 kg slides without friction across smooth ice at an initial velocity $\vec{\mathbf{u}}_1 = 0.8\hat{\mathbf{i}}\,\text{m/s}$. A second block of mass m_2=1.2 kg is initially at rest, and has a spring attached to it of length l=0.25 m, and spring constant k=120 N/m. The free end of the spring is covered with super-glue. The first block bumps head-on into the end of the spring and sticks.

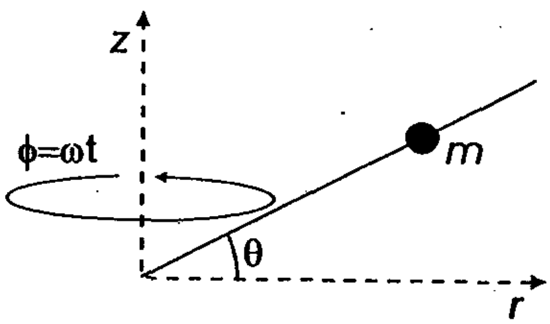


a) [2] What is the velocity of the center of mass of the two blocks after the collision?

b) [5] Assume that the total mechanical energy is conserved in the collision. Some of that energy goes into kinetic energy of the center of mass motion, and some goes into simple harmonic motion. What are the amplitude and period of that simple harmonic motion? Hint: Use the reduced mass to calculate the period.

c) [4] What is the instantaneous velocity of the first block (in the lab frame) exactly one half-period of oscillation after the collision? Assume there is no damping.

2. [5] A bead of mass m slides without friction on a wire sloped at an angle θ with respect to the horizontal. The wire rotates around a vertical axis at constant angular velocity ω, as shown in the figure.



a) [1] Draw a free-body diagram showing all the forces on the bead.

b) [4] Using Newton's 2nd Law, find the position of the bead with $r \neq 0$ such that it doesn't slide up or down the wire. Show clearly what coordinate system you are using. For example, you might choose the standard x-y system with $x=r$ and $y=z$, or you might choose a tilted coordinate system. Either way works fine if you are careful with vector components.

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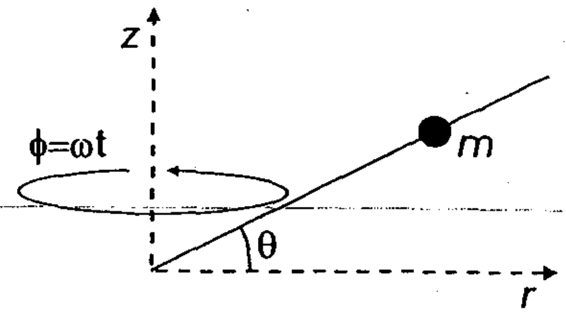
3. [14] Now address the previous problem using Lagrangian mechanics. For simplicity, let the equation describing the wire be $z = Cr$. where C is a constant.



~~a) [3] Write down an expression for the kinetic~~ energy T of the bead, using cylindrical coordinates r, ϕ, and z. Then substitute in the constraints on ϕ and z to express T in terms of the single dynamical variable r.

b) [2] Write down an expression for the potential energy U and the Lagrangian L in terms of the single dynamical variable r.

c) [2] Use Lagrange's equation to get a differential equation for r.

Let's check your work so far. You should now have a differential equation of the form

$$\ddot{r} - ar = -b$$

where a and b are positive constants. If you did parts a-c, then you have expressions for a and b. If not, just continue on from here.)

d) [2] Find the steady-state solution to your equation, with $\dot{r} = 0$ and $\ddot{r} = 0$. Call your solution $r = r_p$. If you have expressions for a and b, compare r_p with the answer you obtained in problem 2b, using the equivalence $C = \tan(\theta)$.

e) [2] Write down a general solution for $r(t)$. This should consist of a solution to the homogeneous differential equation, with two unknown constants A and B, and a particular solution to the inhomogeneous equation. Note that the particular solution is just $r = r_p$.

f) [3] Find the constants A and B and describe the behavior of the bead at all times if the initial conditions are $\dot{r} = 0$ and $r(0) = r_0 \neq r_p$. Discuss the two cases $r_0 < r_p$ and $r_0 > r_p$ separately. From this analysis, say whether or not the solution $r = r_p$ is stable. Could you have figured this out just from looking at the differential equation?

Solutions to Practice Final Exams

Exam 1, Problem 1

$m_1 = 0.4$ kg $\quad m_2 = 1.2$ kg $\quad \vec{u}_1 = 0.8\,\hat{i}$ m/s

(a) What is $\vec{V}_{cm}$?

$$m_1 u_1 + m_2 \cdot 0 = (m_1 + m_2) V_{cm}$$

$$\Rightarrow \vec{V}_{cm} = \frac{m_1}{m_1+m_2}\vec{u}_1 = \frac{0.4}{1.6} \cdot 0.8 = \boxed{0.2\,\hat{i} \text{ m/s}}$$

(b) In SHM $\quad \omega = \sqrt{\frac{K}{\mu}}$ and $\tau = \frac{2\pi}{\omega}$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{0.4 \cdot 1.2}{1.6} = 0.3 \text{ kg}$$

If $K = 120$ N/m $\Rightarrow \tau = 2\pi\sqrt{\frac{\mu}{K}}$

$$\therefore \tau = 2\pi\sqrt{\frac{0.3}{120}} = \boxed{0.314 \text{ sec}}$$

If energy is conserved :- $\quad E_i = E_f = \frac{1}{2}(m_1+m_2)V_{cm}^2 + \frac{1}{2}KA^2$

$$\therefore \frac{1}{2} \cdot 0.4 \cdot 0.8^2 = \frac{1}{2} \cdot 1.6 \cdot 0.2^2 + \frac{1}{2} \cdot 120 \cdot A^2$$

$$\therefore 60 A^2 = 0.128 - 0.032 = 0.096$$

$$\therefore \boxed{A = 0.04 \text{ m}}$$

(c) The velocity of m_1 in the center of mass initially is

$$u_1' = u_1 - V_{cm} = 0.8 - 0.2 = +0.6 \hat{\imath} \text{ m/s}$$

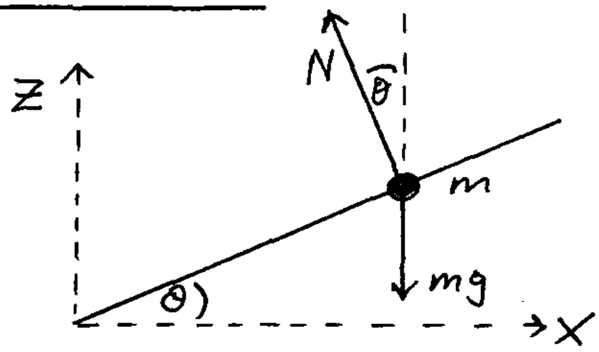
Exactly ½ period later it will have reversed in the CM i.e.

$$v_1' = -0.6 \hat{\imath} \text{ m/s}$$

In the lab its velocity will be

$$v_1 = v_1' + V_{cm} = -0.6 + 0.2 = \boxed{-0.4 \hat{\imath} \text{ m/s}}$$

a)



(b) $\vec{F} = m\vec{a} \Rightarrow$

x direction : $-N\sin\theta = -\dfrac{mv^2}{r} = -mr\omega^2$

z direction : $N\cos\theta - mg = 0 \Rightarrow N = \dfrac{mg}{\cos\theta}$

Substitute into the x equation $\Rightarrow$

$$mg\tan\theta = mr\omega^2$$

$$\therefore \boxed{r = g\frac{\tan\theta}{\omega^2}}$$

(a) Kinetic energy $T = \frac{1}{2} m v^2$

In cylindrical coordinates $\Rightarrow$ $T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2 + \dot{z}^2)$

We have $\phi = \omega t \Rightarrow \dot{\phi} = \omega$ and $z = Cr \Rightarrow \dot{z} = C\dot{r}$

$$\therefore \quad T = \frac{1}{2} m (\dot{r}^2 + C^2 \dot{r}^2 + \omega^2 r^2)$$

(b) $U = mgz = mgCr$

$$L = T - U = \frac{1}{2} m \dot{r}^2 (1 + C^2) + \frac{1}{2} m \omega^2 r^2 - mgCr$$

(c) $\frac{\partial L}{\partial r} = m\omega^2 r - mgC \qquad \frac{\partial L}{\partial \dot{r}} = m\dot{r}(1 + C^2)$

So Lagrange's equation $\Rightarrow$

$$\cancel{m}\omega^2 r - \cancel{m}gC - \cancel{m}\ddot{r}(1 + C^2) = 0$$

$$\therefore \boxed{\ddot{r} - \frac{\omega^2}{(1+C^2)} r = -\frac{gC}{(1+C^2)}}$$

(d) If $\ddot{r} = \dot{r} = 0 \Rightarrow \frac{\omega^2 r_p}{(1+C^2)} = \frac{gC}{(1+C^2)}$

$$\therefore \boxed{r_p = \frac{gC}{\omega^2} = g\frac{\tan\theta}{\omega^2}}$$

As in problem 2

(e) The homogeneous equation is $\ddot{r} - \frac{\omega^2}{(1+c^2)} r$

which looks like SHM so I write the general solution as

$$r(t) = Ae^{\alpha t} + Be^{-\alpha t} + r_p \qquad \text{with } \alpha = \frac{\omega}{\sqrt{1+c^2}}$$

(f) $\dot{r}(t) = \alpha A e^{\alpha t} - \alpha B e^{-\alpha t}$

If $\dot{r}(0) = 0 \quad \Rightarrow \quad A = B$

$$\therefore r(t) = A\left(e^{\alpha t} + e^{-\alpha t}\right) + r_p$$

If $r(0) = r_0 = 2A + r_p \quad \therefore \boxed{A = \frac{r_0 - r_p}{2}}$

If $r_0 < r_p \quad \Rightarrow \quad A < 0$ and $r(t)$ decreases with t

If $r_0 > r_p \quad \Rightarrow \quad A > 0$ and $r(t)$ increases with t

$\Rightarrow r = r_p$ is <u>not</u> stable.

(It looked like SHM but it wasn't!)

PHYSICS 321 - Classical Mechanics I

Two Practice Exams! **Practice Exam #2**

Total points = 30. There are only 3 problems on this exam. Think about what is going on in each problem, and show all your work. Include units in your answers.

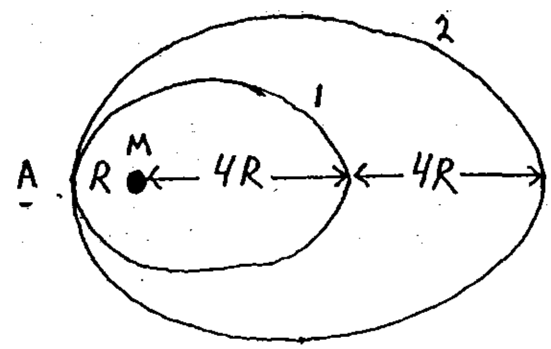
1. [10] A particle of mass m_1=5.0 kg traveling at an initial velocity u_1 =12.0 m/s in the positive x direction collides with a stationary particle of mass m_2=15.0 kg. After the collision, the first particle is traveling with speed v_1 = 7.16 m/s at an angle ψ = 34.4° with respect to the x direction.

   a) [4] What are the final speed and direction of the second particle? (Find v_2 and ζ.) Warning: do not assume that the collision is elastic; it is not.

   b) [2] What is the velocity of the center of mass frame with respect to the lab? Draw a picture of the situation in the center-of-mass frame before the collision, showing clearly the initial velocities u_1' and u_2' of the two masses.

   c) [4] Draw a picture of the situation in the center-of-mass frame after the collision. What are the final speeds v_1' and v_2' of the two particles in the CM frame? What is the scattering angle θ in the center of mass frame? (Calculate these quantities by transforming the final velocities from the lab to the CM frame – not by memorizing complicated formulas.)

2. [5] A spaceship of mass m is orbiting a planet of mass $M >> m$ in the elliptical orbit labeled 1 in the diagram. The astronauts want to change to the elliptical orbit labeled 2. When they reach point A, by how much should they change their velocity? Express your answer in terms of G, M, and R. (To do this problem, calculate the velocity of the spaceship at point A both before and after the change.)



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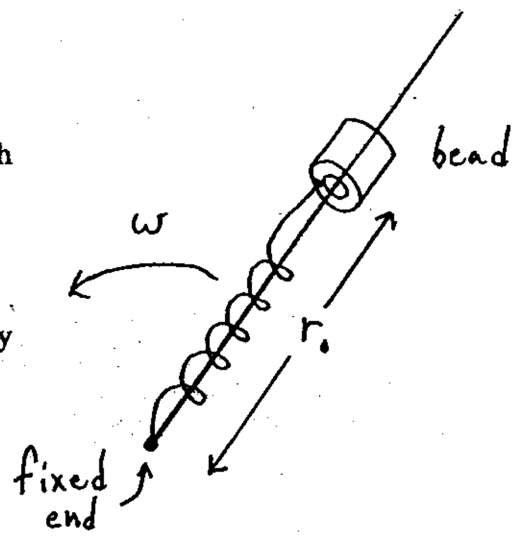
3. [15] A bead of mass m slides without friction on a horizontal wire. A spring of stiffness k and rest length r_0 is attached to the bead and to one end of the wire. As you know, if we displace the bead from $r = r_0$ and let go, it will oscillate with frequency $\omega_0=\sqrt{k/m}$.



Suddenly the wire starts rotating with angular velocity $\omega<\omega_0$ around the end where the spring is fixed. The rotation is in a horizontal plane, so gravity plays no role in this problem. We'll do this problem using Lagrangian mechanics, one step at a time.

a) [3] Write down an expression for the kinetic energy T of the bead, using plane polar coordinates r and θ. Then substitute in the constraint on θ so you can express T in terms of the single dynamical variable r.

b) [2] Write down an expression for the potential energy U. Remember that the rest length of the spring is r_0, not zero. (That means that the potential energy is zero when $r = r_0$.)

c) [3] Write down the Langrangian in terms of the single dynamical variable r. Use Lagrange's equation to get a differential equation for r.

Let's check your work so far. You should now have a differential equation of the form

$$\ddot{r} + ar = b$$

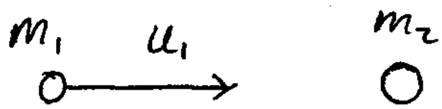
where a and b are positive constants. If you did parts a-c, then you should have expressions for a and b in terms of r_0, ω and ω_0. If not, just continue on from here. (If your expression for a looks worrisome, remember that $\omega<\omega_0$.)

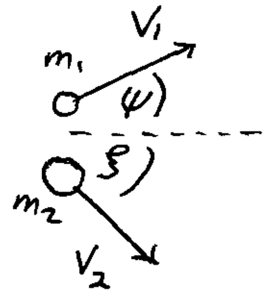
d) [2] Find the steady-state solution to your equation, with $\dot{r} = 0$ and $\ddot{r} = 0$. Call your solution $r = r_p$.

e) [2] Write down the general solution for $r(t)$. This should consist of a solution to the homogeneous differential equation, with two unknown constants A and B, and a particular solution to the inhomogeneous equation. The particular solution is just $r = r_p$.

f) [3] Find the constants A and B if the initial conditions are $\dot{r}(0) = 0$ and $r(0) = r_0$. Describe in words the behavior of the bead at all times $t>0$. (Note that $r_0 < r_p$.)

Exam 2, Problem 1





we are given
$$m_1 = 5 \text{ kg}$$
$$m_2 = 15 \text{ kg}$$
$$u_1 = 12.0 \text{ m/s}$$
$$V_1 = 7.16 \text{ m/s}$$
$$\psi = 34.4^\circ$$

(a) Using conservation of momentum $\vec{P_i} = \vec{P_f}$

In the x direction: $m_1 u_1 + 0 = m_1 V_1 \cos\psi + m_2 V_2 \cos\xi$

$$\therefore V_2 \cos\xi = m_1 u_1 - m_1 V_1 \cos\psi$$

$$= \frac{5 \cdot 12 - 5 \cdot 7.16 \cdot \cos 34.4}{15}$$

$$= \frac{1}{3}(12 - 5.91) = 2.03 \text{ m/s}$$

In the y direction: $0 = m_1 V_1 \sin\psi - m_2 V_2 \sin\xi$

$$\therefore V_2 \sin\xi = \frac{m_1}{m_2} V_1 \sin\psi = \frac{1}{3} \cdot 7.16 \cdot \sin 34.4$$

$$= 1.35 \text{ m/s}$$

$$\Rightarrow \quad \frac{V_2 \sin \xi}{V_2 \cos \xi} = \tan \xi = \frac{1.35}{2.03} = 0.664$$

$$\therefore \boxed{\xi = 33.6^\circ}$$

Then $V_2 \sin 33.6 = 1.35$

$$\therefore \boxed{V_2 = 2.44 \ m/s}$$

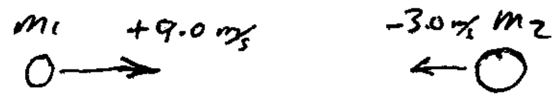
(b) $$V_{cm} = \frac{m_1 u_1}{m_1 + m_2} = \frac{u_1}{4} = \boxed{3.0 \ m/s}$$

in the positive x direction

In the CM frame before the collision

$$u_1' = u_1 - V_{cm} = 12 - 3 = 9.0 \ m/s$$

$$u_2' = u_2 - V_{cm} = 0 - 3 = -3.0 \ m/s$$



(c) In the CM frame after the collision

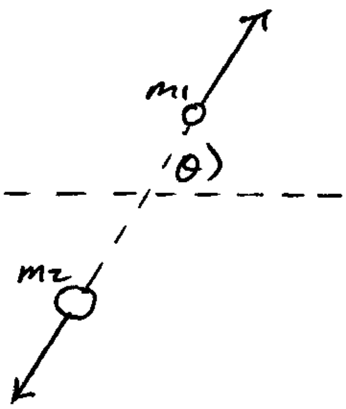
$$\vec{V}_1' = \vec{V}_1 - \vec{V}_{cm}$$

In the x direction: $V_{1x}' = V_1 \cos\psi - V_{cm}$

$= 5.91 - 3 = 2.91 \text{ m/s}$

In the y direction: $V_{1y}' = V_1 \sin\psi = 7.16 \cdot \sin 34.4$

$= 4.03 \text{ m/s}$



$$\tan\theta = \frac{V_{1y}'}{V_{1x}'} = \frac{4.03}{2.91} = 1.38$$

$$\Rightarrow \boxed{\theta = 54.2^\circ}$$

$$V_1' = \sqrt{V_{1x}'^2 + V_{1y}'^2} = \sqrt{2.91^2 + 4.03^2}$$

$$\boxed{V_1' = 4.97 \text{ m/s}}$$

Since $\vec{p}' = 0$ in the CM frame $m_1 V_1' = m_2 V_2'$

$$\therefore V_2' = \frac{1}{3} \cdot 4.97 = \boxed{1.66 \text{ m/s}}$$

For orbit 1 $2a_1 = 5R$ For orbit 2 $2a_2 = 9R$

At point A $r_A = R$ Use $k = GMm$

Total energy of an ellipse is $E = -\frac{k}{2a}$

$\therefore$ For orbit 1 : $-\frac{k}{5R} = -\frac{k}{R} + \frac{1}{2} m v_1^2$

$$\therefore v_1^2 = \frac{2}{m} k \left(\frac{1}{R} - \frac{1}{5R} \right) = \frac{2GM}{R} \left(1 - \frac{1}{5} \right) = \frac{8}{5} \frac{GM}{R}$$

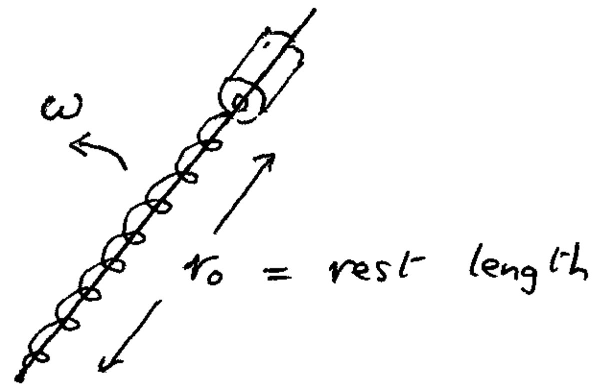
For orbit 2 :

$$v_2^2 = \frac{2}{m} k \left(\frac{1}{R} - \frac{1}{9R} \right) = \frac{2GM}{R} \left(1 - \frac{1}{9} \right) = \frac{16}{9} \frac{GM}{R}$$

$$\Delta v = v_2 - v_1 = \sqrt{\frac{16}{9} \frac{GM}{R}} - \sqrt{\frac{8}{5} \frac{GM}{R}}$$

$$= \left(\frac{4}{3} - \sqrt{\frac{8}{5}} \right) \sqrt{\frac{GM}{R}} = \boxed{0.068 \sqrt{\frac{GM}{R}}}$$

Positive, so Δv is in same direction as $\vec{v}_1$.



(a) $T = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$

$\theta = \omega t \Rightarrow \dot{\theta} = \omega$

$\therefore \boxed{T = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m \omega^2 r^2}$

(b) $\boxed{U = \frac{1}{2} k (r - r_0)^2}$

(c) $L = T - U = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m \omega^2 r^2 - \frac{1}{2} k (r - r_0)^2$

$\frac{\partial L}{\partial r} = m\omega^2 r - k(r - r_0) \qquad \frac{\partial L}{\partial \dot{r}} = m\dot{r}$

$\Rightarrow$ Lagrange's eqt is

$$m\omega^2 r - k(r - r_0) - m\ddot{r} = 0$$

$$\therefore\ \ddot{r} + \frac{k}{m}(r - r_0) - \omega^2 r = 0$$

Subst $\omega_0^2 = \frac{k}{m}$ $\Rightarrow$

$$\ddot{r} + r\omega_0^2 - r_0\omega_0^2 - \omega^2 r = 0$$

or $\boxed{\ddot{r} + (\omega_0^2 - \omega^2) r = \omega_0^2 r_0}$

(d) If $\ddot{r} = 0$ $\Rightarrow$ $\boxed{r_p = \frac{\omega_0^2\, r_0}{\omega_0^2 - \omega^2} = \frac{r_0}{1 - \frac{\omega^2}{\omega_0^2}}}$

e) General solution is

$$r(t) = A\cos\omega_1 t + B\sin\omega_1 t + r_p \qquad \text{with } \omega_1 = \sqrt{\omega_0^2 - \omega^2}$$

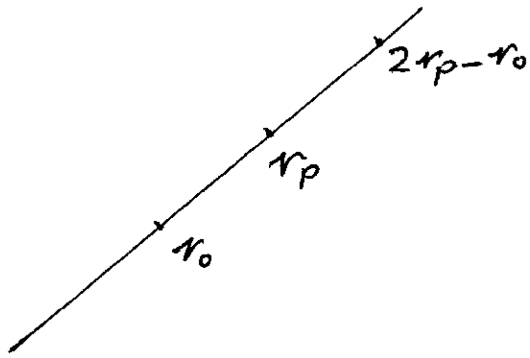
$$\dot{r}(t) = -\omega_1 A\sin\omega_1 t + \omega_1 B\cos\omega_1 t$$

If $\dot{r}(0) = 0$ $\Rightarrow$ $B = 0$

If $r(0) = r_0$ $\Rightarrow$ $r_0 = A + r_p$ $\therefore$ $A = r_0 - r_p$

So solution is $\boxed{r(t) = (r_0 - r_p)\cos\omega_1 t + r_p}$

with $r_p = \frac{r_0}{1 - \frac{\omega^2}{\omega_0^2}}$



The bead oscillates between r_0 and $2r_p - r_0$ with an angular frequency of $\omega_1 = \sqrt{\omega_0^2 - \omega^2}$