PHY492: Nuclear & Particle Physics

Lecture 2

Applications of Relativistic Kinematics

Rutherford scattering

A particle's total energy, momentum, and mass

Momentum and rest energy

$$p = \gamma m v$$

$$pc = \gamma m v c = \gamma\beta m c^2$$

particle's beta and gamma

$$\beta = \frac{v}{c}\ , \ \gamma = \left(1-\beta^2\right)^{-\frac{1}{2}}$$

Total energy and rest energy

$$E^2 = p^2c^2 + m^2c^4 = \gamma^2\beta^2 m^2c^4 + m^2c^4$$

$$E = \left(1+\gamma^2\beta^2\right)^{\frac{1}{2}} mc^2 = \gamma mc^2$$

surprisingly

$$\left(1+\gamma^2\beta^2\right)^{\frac{1}{2}} = \left(1+\frac{\beta^2}{1-\beta^2}\right)^{\frac{1}{2}} = \left(\frac{1}{1-\beta^2}\right)^{\frac{1}{2}} = \gamma$$

Simplicity

$$\gamma = \frac{E}{mc^2}; \quad \beta = \frac{pc}{E}$$

$$T = E - mc^2 = \left(\gamma - 1\right)mc^2$$

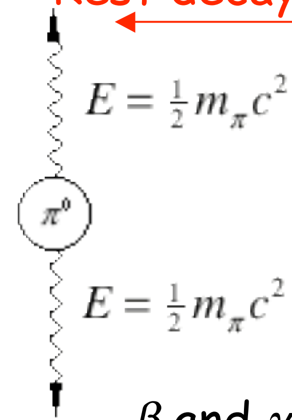
Short Hand(without the c's)

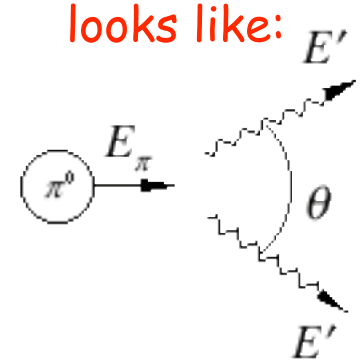
$$\gamma = \frac{E}{m}; \quad \left(\gamma - 1\right) = \frac{T}{m}; \quad \beta = \frac{p}{E};$$

Example (HEP) using a moving frame

A π^0 at rest decays to two photons along the y direction, each with an energy equal to *half the rest energy* of the π^0 . If this π^0 decays while moving in the z direction with an energy $E_\pi >> m_\pi c^2$, what is the opening angle θ, of the two photons?

Rest decay in moving frame looks like:





Photon E and p in rest frame

$$m_\gamma = 0, \quad E_\gamma = p_\gamma c; \quad p_y c = \tfrac{1}{2} m_\pi c^2; \quad p_z c = 0$$

Lorentz transform photon E and p

$$p'_y c = p_y c = \tfrac{1}{2} m_\pi c^2$$

$$p'_z c = \gamma\left(p_z c - \beta E\right) = \gamma E = \frac{E_\pi}{m_\pi c^2}\left(\tfrac{1}{2} m_\pi c^2\right) = \tfrac{1}{2} E_\pi$$

β and γ of Lorentz trans.

$$\gamma = \gamma_\pi = \frac{E_\pi}{m_\pi c^2}$$

$$\beta = -\beta_\pi = -\frac{p_\pi c}{E_\pi} \approx -1$$

Trigonometry & small angle approx.

$$\tan\frac{\theta}{2} = \frac{p'_y c}{p'_z c} = \frac{m_\pi c^2}{E_\pi} << 1$$

$$\theta \approx \frac{2 m_\pi c^2}{E_\pi}$$

$$\text{if } E_\pi = 20 \text{ GeV}; \quad m_\pi c^2 = 140 \text{ MeV}$$

$$\theta = \frac{2\left(140 \text{ MeV}\right)}{20{,}000 \text{ MeV}} = 14 \text{ mr} = 0.8^\circ$$

Example (NP)

One of the reactions by which energy is liberated in the sun is:

$$ {}^{1}\mathrm{H}_1 + {}^{2}\mathrm{H}_1 \rightarrow {}^{3}\mathrm{He}_2 + \gamma + \text{energy}. $$

Given the masses of the nuclei, and assuming the reaction takes place at rest, what are the (kinetic) energy and momentum of the reaction products?

Masses are ${}^{1}\mathrm{H} : 938.3\ \mathrm{MeV/c^2};\quad {}^{2}\mathrm{H} : 1875.7\ \mathrm{MeV/c^2};\quad {}^{3}\mathrm{He} : 2808.5\ \mathrm{MeV/c^2}$

Energy released in the reaction: $Q = (1875.7 + 938.3 - 2808.5)c^2 = 5.5\ \mathrm{MeV}$

Energy & Momentum Conservation

$$ T_{He} + E_\gamma = 5.5\ \mathrm{MeV} $$

$$ p_{He} = p_\gamma = \frac{E_\gamma}{c} $$

$$ T_{He} = \frac{p_{He}^2 c^2}{2m_{He}c^2} = \frac{E_\gamma^2}{2(2808.5\ \mathrm{MeV})} \approx 0 $$

$$ T_{He} << E_\gamma $$

$$ E_\gamma = 5.5\ \mathrm{MeV} $$

$$ p_{He} = p_\gamma = 5.5\ \mathrm{MeV/c} $$

$$ T_{He} = \frac{(5.5)^2}{2(2808.5)}\mathrm{MeV} = .005\ \mathrm{MeV} = 5\ \mathrm{keV} $$

Example (on the border between NP & HEP)

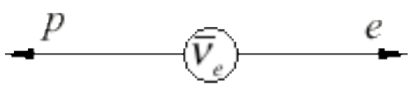
On average, a free neutron will last about 13 minutes before decaying to a proton + electron + antineutrino, a reaction also known as neutron β-decay.

$n \to p + e + \bar{\nu}_e$ Energy released: $Q = (m_n - m_p - m_e)c^2 = 0.8\ \text{MeV}$

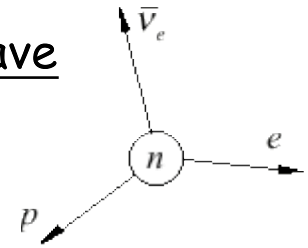
Given the masses of the proton and electron (neutrinos have very, very small masses), determine the "end-point energy" of the electron.

$m_n = 939.6\ \text{MeV/c}^2 \quad m_p = 938.3\ \text{MeV/c}^2 \quad m_e = 0.51\ \text{MeV/c}^2$

Electron has max. energy (why?)



e does not have max. energy



Energy & Momentum Conservation

$$p_p = p_e$$
$$T_e + T_p = 0.8\ \text{MeV}$$

$$\text{proton N.R.} \quad T_p = \frac{p_p^2 c^2}{2m_p c^2} = \frac{p_e^2 c^2}{2m_p c^2} << T_e$$

proton energy 0.1% of electron energy.

$$T_e = 0.8\ \text{MeV}$$

electron takes nearly all the energy

$$p_e^2 c^2 = T_e^2 + 2m_e c^2 T_e$$
$$T_p = \frac{Q(Q + 2m_e c^2)}{2m_p c^2} = \frac{0.8(0.8 + 1.0)}{1876}\ \text{MeV} = 0.8\ \text{keV}$$

Application for invariants in HEP

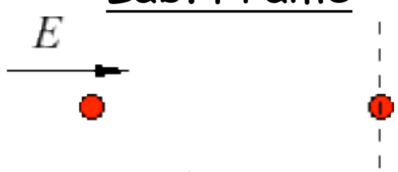
Add "4 vectors" of two particles yields a new "4 vector"

$$\left[E_1, \vec{p}_1 c\right] + \left[E_2, \vec{p}_2 c\right] = \left[\left(E_1 + E_2\right), \left(\vec{p}_1 + \vec{p}_2\right) c\right]$$

and another invariant $\left(E_1 + E_2\right)^2 - \left(\vec{p}_1 + \vec{p}_2\right)^2 c^2 = m_{12}^2 c^4$ effective mass² of pair

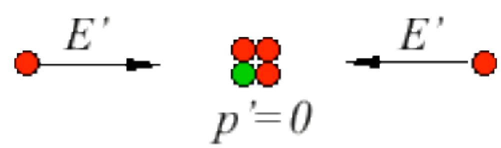
A beam of protons hits a Hydrogen target. What is the minimum energy of the beam needed to create a proton-antiproton pair?

Lab. Frame



$$E_T = mc^2$$
$$p_T = 0$$

Center of Mass Frame



invariant for system

$$s = \left(E_B + E_T\right)^2 - \left(p_B + p_T\right)^2 c^2$$
$$= \left(E + mc^2\right)^2 - \left(E^2 - m^2 c^4\right)$$
$$= 2mc^2 \left(E + mc^2\right)$$

Minimum energy will leave all particles (3 p, 1 pbar) at rest in the center of mass

same invariant

$$s = \left(4mc^2\right)^2$$

$$2mc^2 \left(E + mc^2\right) = 16 m^2 c^4$$
$$E = 7mc^2$$

In lab frame each "proton" has the some momentum

Rutherford and alpha rays from Radium

Measured particle emission rate & charge

Fig. (A)

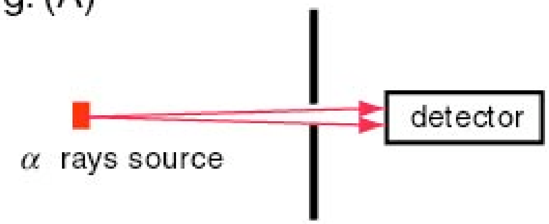
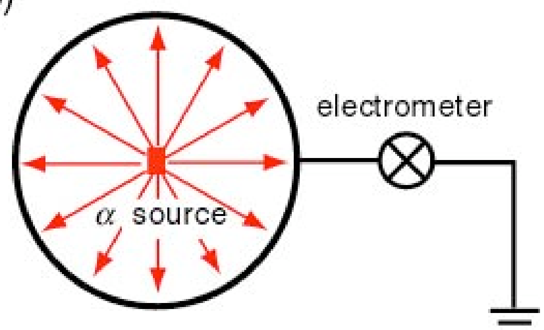


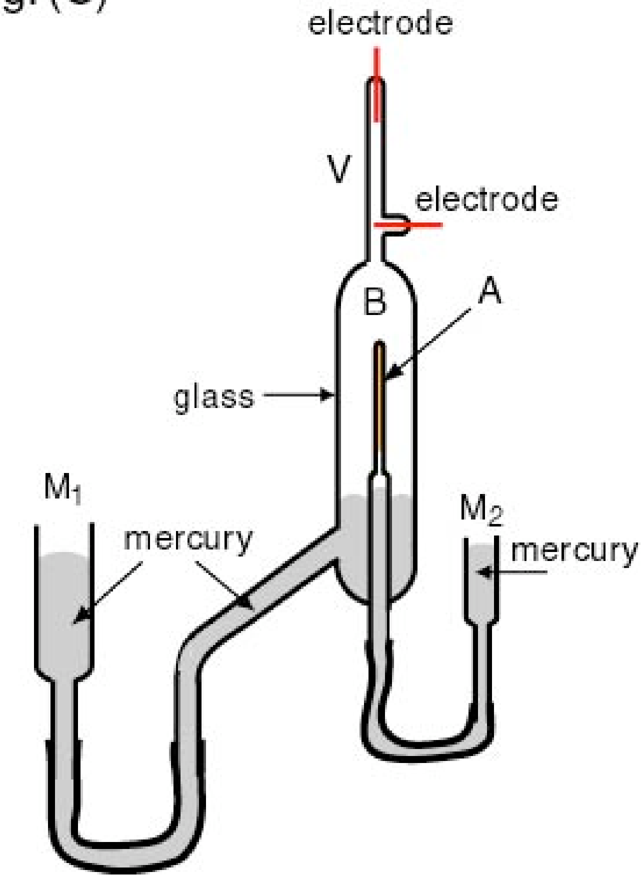
Fig. (B)



Also measured q/m = 1/2 that of Hydrogen ions.

Measured spectrum is that of Helium

Fig. (C)



http://www2.kutl.kyushu-u.ac.jp/seminar/MicroWorld1_E/

Scattering by thin gold foils (on glass)

Measured α scattering through materials

Rutherford and Geiger first measured scattering of a particles through the foils. With the thinnest foils, scattering was only 1 or 2 degrees.

Undergraduate experiment for Ernest Marsden: "See if any alpha particles are scattered backward."

To Rutherford's great surprise, there were indeed alphas scattered backward. He later said: "It was like firing a 15" shell at a piece of paper and having it bounce back".

Fig. (D) Scattering of α rays by an atom

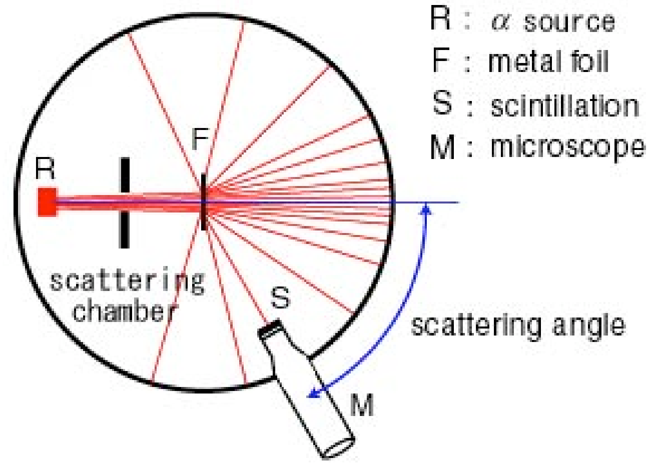
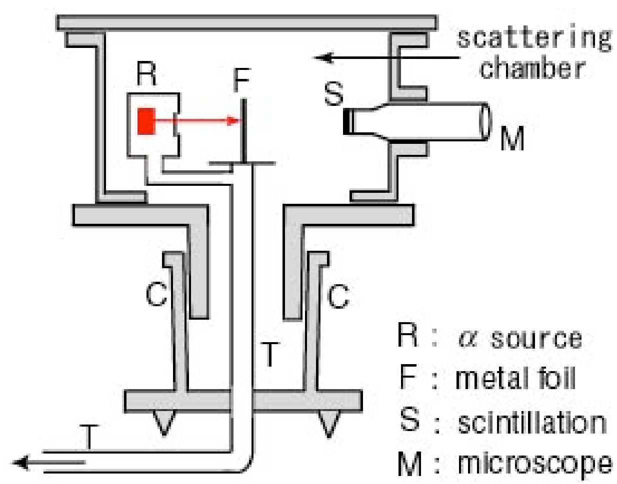


Fig. (E) Setting of the experiment



Comments on Units

Das & Ferbel use the c.g.s. unit system

$$F_E = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2} = \left(9\times 10^9 \ \frac{\mathrm{Nm}^2}{\mathrm{C}^2}\right)\frac{q_1 q_2}{r^2}$$

$$\frac{1}{4\pi\varepsilon_0} = \left(9\times 10^9 \ \frac{\mathrm{Nm}^2}{\mathrm{C}^2}\right)\left(\frac{1\times 10^5 \,\mathrm{dyne}}{1\mathrm{N}}\right)\left(\frac{1\times 10^4 \,\mathrm{cm}^2}{1 \ \mathrm{m}^2}\right)$$

$$= \left(3\times 10^9\right)^2 \frac{\mathrm{dyne}\bullet\mathrm{cm}^2}{\mathrm{C}^2}$$

Let $3\times 10^9 \ \mathrm{esu} = 1 \ \mathrm{C}$

$$= 1 \ \frac{\mathrm{dyne}\bullet\mathrm{cm}^2}{\mathrm{esu}^2}$$

$$e = 1.6\times 10^{-19}\mathrm{C}\left(3\times 10^9 \ \mathrm{esu/C}\right)$$
$$= 4.8\times 10^{-10} \ \mathrm{esu}$$

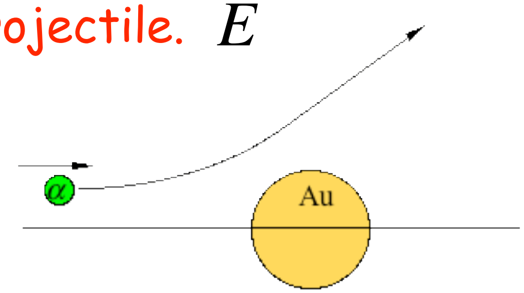
$$\vec{F}_M = q\vec{v}\times\vec{B} \quad F(\mathrm{N}),\ q(\mathrm{C}),\ v(\mathrm{m/s}),\ B(\mathrm{T})$$

$$= q\vec{v}\times\frac{\vec{B}}{c} \quad F(\mathrm{dyne}),\ q(\mathrm{esu}),\ v(\mathrm{cm/s}),\ B(\mathrm{G})$$

Rutherford scattering

α scattering off Gold foil

Kinetic energy of projectile. E



Charges of projectile and target. $Ze, Z'e$

Cross section per unit solid angle as a function of the scattering angle, θ

$$\frac{d\sigma}{d\Omega}(\theta) = \left(\frac{ZZ'e^2}{4E}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}}$$

Use fine structure constant to simply calculations

$$\frac{ZZ'e^2}{4E} = \frac{ZZ'e^2(\hbar c)}{4E(\hbar c)} = \frac{ZZ'}{4E}\alpha\hbar c$$

$$= \frac{ZZ'}{4E}\frac{197\ \text{MeV}\bullet\text{fm}}{137}$$

Fine Structure Constant

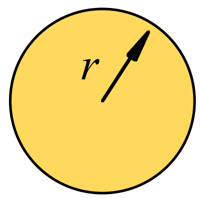
$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}\ (\text{SI}) = \frac{e^2}{\hbar c}\ (\text{c.g.s}) = \frac{1}{137}$$

$$\hbar c = 197\ \text{MeV}\bullet\text{fm}$$

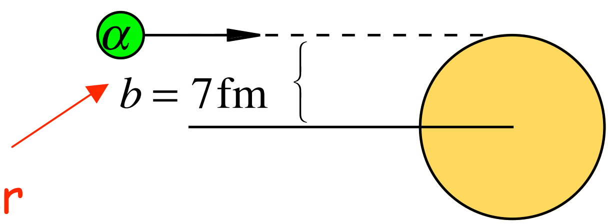
$$1\ \text{fm} = 1\times10^{-15}\ \text{m}$$

Impact parameter in Rutherford scattering

<u>Scattering from Gold nucleus</u>



Gold nucleus
Z=79, A=197



Impact parameter

D&F Eq. 1.32

$$b = \frac{ZZ'e^2}{2E}\cot\frac{\theta}{2}$$

<u>"Cross sectional" area of a Gold nucleus</u>

$$b = 7\text{fm}$$

$$r = 7\text{fm} = 7\times 10^{-15}$$

$$A = \pi r^2 = 154\ \text{fm}^2 = 1.54\times 10^{-28}\ \text{m}^2$$

$$= 1.54\ \text{barns}$$

$$1\ \text{barn} = 1\times 10^{-28}\ \text{m}^2 = (10\ \text{fm})^2$$

<u>Small b, large θ</u>

<u>Large b, small θ</u>

http://hyperphysics.phy-astr.gsu.edu/hphys.html
http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/crosec.html

Total Cross Section calculation

Rutherford scattering

$$\frac{d\sigma}{d\Omega}(\theta) = \left(\frac{ZZ'e^2}{4E}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}}$$

Trigonometry

$$d\left(\sin\frac{\theta}{2}\right) = \frac{1}{2}\cos\frac{\theta}{2}\,d\theta$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\sin\theta\, d\theta = 4\sin\frac{\theta}{2}\, d\left(\sin\frac{\theta}{2}\right)$$

See D&F deriv. of Eq. 1.32

$$\sin^2\frac{\theta_b}{2} = \frac{1}{1+\frac{4b^2E^2}{\left(ZZ'e^2\right)^2}}$$

Total Cross Section

$$\sigma_{TOTAL} = \int \frac{d\sigma}{d\Omega}(\theta)\,d\Omega \quad \text{where } d\Omega = \sin\theta\, d\theta\, d\phi$$

$$= \left(\frac{ZZ'e^2}{4E}\right)^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \frac{1}{\sin^4\frac{\theta}{2}}$$

$$= 8\pi\left(\frac{ZZ'e^2}{4E}\right)^2 \int_0^1 d\left(\sin\frac{\theta}{2}\right)\frac{1}{\sin^3\frac{\theta}{2}}$$

Diverges!

$$\sigma_{TOT}\left(\theta > \theta_b\right) = 8\pi\left(\frac{ZZ'e^2}{4E}\right)^2 \int_{\sin\frac{\theta_b}{2}}^{1} \frac{dx}{x^3}$$

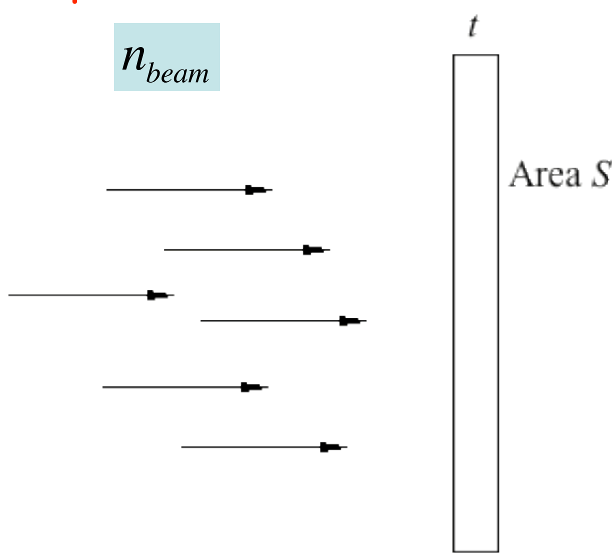
$$= \pi b^2$$

Area!

Measuring cross sections in thin targets

beam particles/s

n_{beam}



$$m_T = \rho t S$$ mass of target

$$n_{moles} = \frac{m_T}{A}$$ # of target moles
atomic mass A

$$n_{nuclei} = n_{moles} A_0$$ # target nuclei

$$\frac{n_{nuclei}}{S}$$ # target nuclei
per unit area

scattered particles
detected per second

$$n_{\text{detected}} = n_{beam} \frac{n_{nuclei}}{S} \frac{d\sigma}{d\Omega}(\theta,\phi) d\Omega_{\text{de tector}}$$