Lecture 2
Applications of Relativistic Kinematics
Rutherford scattering
A particle’s total energy, momentum, and mass

**Momentum and rest energy**

\[ p = \gamma mv \]
\[ pc = \gamma mvc = \gamma \beta mc^2 \]

**Total energy and rest energy**

\[ E^2 = p^2 c^2 + m^2 c^4 = \gamma^2 \beta^2 m^2 c^4 + m^2 c^4 \]
\[ E = \left(1 + \gamma^2 \beta^2\right)^{\frac{1}{2}} mc^2 = \gamma mc^2 \]

**Simplicity**

\[ \gamma = \frac{E}{mc^2}; \quad \beta = \frac{pc}{E} \]
\[ T = E - mc^2 = (\gamma - 1)mc^2 \]

**Particle’s beta and gamma**

\[ \beta = \frac{v}{c}, \quad \gamma = \left(1 - \beta^2\right)^{-\frac{1}{2}} \]

**Surprisingly**

\[ \left(1 + \gamma^2 \beta^2\right)^{\frac{1}{2}} = \left(1 + \frac{\beta^2}{1 - \beta^2}\right)^{\frac{1}{2}} = \left(\frac{1}{1 - \beta^2}\right)^{\frac{1}{2}} = \gamma \]

**Short Hand (without the c’s)**

\[ \gamma = \frac{E}{m}; \quad (\gamma - 1) = \frac{T}{m}; \quad \beta = \frac{p}{E}; \]
Example (HEP) using a moving frame

A $\pi^0$ at rest decays to two photons along the $y$ direction, each with an energy equal to half the rest energy of the $\pi^0$. If this $\pi^0$ decays while moving in the $z$ direction with an energy $E_\pi \gg m_\pi c^2$, what is the opening angle $\theta$, of the two photons?

Photon $E$ and $p$ in rest frame

- $m_\gamma = 0$, $E_\gamma = p_\gamma c$, $p_\gamma c = \frac{1}{2} m_\pi c^2$; $p_z c = 0$

Lorentz transform photon $E$ and $p$

- $p'_y c = p_y c = \frac{1}{2} m_\pi c^2$
- $p'_z c = \gamma (p_z c - \beta E) = \gamma E = \frac{E_\pi}{m_\pi c^2} \left( \frac{1}{2} m_\pi c^2 \right) = \frac{1}{2} E_\pi$

Trigonometry & small angle approx.

- $\tan \frac{\theta}{2} = \frac{p'_y c}{p'_z c} = \frac{m_\pi c^2}{E_\pi} \ll 1$
- $\theta \approx \frac{2 m_\pi c^2}{E_\pi}$

Rest decay in moving frame looks like:

$\beta$ and $\gamma$ of Lorentz trans.

- $\gamma = \gamma_\pi = \frac{E_\pi}{m_\pi c^2}$
- $\beta = -\beta_\pi = -\frac{p_\pi c}{E_\pi} \approx -1$

if $E_\pi = 20$ GeV; $m_\pi c^2 = 140$ MeV

- $\theta = \frac{2 (140 \text{ MeV})}{20,000 \text{ MeV}} = 14 \text{ mr} = 0.8^\circ$
Example (NP)

One of the reactions by which energy is liberated in the sun is:

\[ ^1H_1 + ^2H_1 \rightarrow ^3He_2 + \gamma + \text{energy}. \]

Given the masses of the nuclei, and assuming the reaction takes place at rest, what are the (kinetic) energy and momentum of the reaction products?

Masses are \(^1H : 938.3 \text{ MeV}/c^2; \quad ^2H : 1875.7 \text{ MeV}/c^2; \quad ^3He : 2808.5 \text{ MeV}/c^2\)

Energy released in the reaction: \[ Q = (1875.7 + 938.3 - 2808.5) c^2 = 5.5 \text{ MeV} \]

Energy & Momentum Conservation

\[
\begin{align*}
T_{He} + E_\gamma &= 5.5 \text{ MeV} \\
p_{He} &= p_\gamma = \frac{E_\gamma}{c}
\end{align*}
\]

\[
\begin{align*}
T_{He} &= \frac{p_{He}^2 c^2}{2m_{He} c^2} = \frac{E_\gamma^2}{2(2808.5 \text{ MeV})} \approx 0 \\
T_{He} &< < E_\gamma
\end{align*}
\]

\[ E_\gamma = 5.5 \text{ MeV} \]

\[ p_{He} = p_\gamma = 5.5 \text{ MeV}/c \]

\[
T_{He} = \frac{(5.5)^2}{2(2808.5)} \text{ MeV} = .005 \text{ MeV} = 5 \text{ keV}
\]
Example (on the border between NP & HEP)

On average, a free neutron will last about 13 minutes before decaying to a proton + electron + antineutrino, a reaction also known as neutron $\beta$-decay.

\[ n \rightarrow p + e + \bar{\nu}_e \]

Energy released: \[ Q = (m_n - m_p - m_e)c^2 = 0.8 \text{ MeV} \]

Given the masses of the proton and electron (neutrinos have very, very small masses), determine the “end-point energy” of the electron.

\[ m_n = 939.6 \text{ MeV}/c^2 \quad m_p = 938.3 \text{ MeV}/c^2 \quad m_e = 0.51 \text{ MeV}/c^2 \]

Electron has max. energy \( \text{(why?)} \)  
\( e \) does not have max. energy

\[ p_p = p_e \]
\[ T_e + T_p = 0.8 \text{ MeV} \]
\[ T_e = 0.8 \text{ MeV} \]

Proton N.R.  \[ T_p = \frac{p_p^2 c^2}{2m_p c^2} = \frac{p_e^2 c^2}{2m_p c^2} \ll T_e \]

Proton energy 0.1% of electron energy.

\[ p_e^2 c^2 = T_e^2 + 2m_e c^2 T_e \]
\[ T_p = \frac{Q(Q + 2m_e c^2)}{2m_p c^2} = \frac{0.8(0.8 + 1.0)}{1876} \text{ MeV} = 0.8 \text{ keV} \]
Application for invariants in HEP

Add “4 vectors” of two particles yields a new “4 vector”

\[
[ E_1, \vec{p}_1 c ] + [ E_2, \vec{p}_2 c ] = \left[ (E_1 + E_2), (\vec{p}_1 + \vec{p}_2) c \right]
\]

and another invariant

\[
(E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 c^2 = m_{12}^2 c^4
\]

A beam of protons hits a Hydrogen target. What is the minimum energy of the beam needed to create a proton-antiproton pair?

**Lab. Frame**

\[
E_T = mc^2
\]

\[
p_T = 0
\]

**Center of Mass Frame**

Minimum energy will leave all particles (3 p, 1 pbar) at rest in the center of mass same invariant

\[
s = \left(4mc^2\right)^2
\]

In lab frame each “proton” has the same momentum

\[
2mc^2 (E + mc^2) = 16m^2 c^4
\]

\[
E = 7mc^2
\]
Rutherford and alpha rays from Radium

Measured particle emission rate & charge

Fig. (A)

Also measured $q/m = 1/2$ that of Hydrogen ions.

Measured spectrum is that of Helium

Fig. (C)

http://www2.kutl.kyushu-u.ac.jp/seminar/MicroWorld1_E/
Scattering by thin gold foils (on glass)

Measured $\alpha$ scattering through materials

Rutherford and Geiger first measured scattering of alpha particles through the foils. With the thinnest foils, scattering was only 1 or 2 degrees.

Undergraduate experiment for Ernest Marsden: “See if any alpha particles are scattered backward.”

To Rutherford’s great surprise, there were indeed alphas scattered backward. He later said: “It was like firing a 15” shell at a piece of paper and having it bounce back.”
Comments on Units

Das & Ferbel use the c.g.s. unit system

\[ F_E = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} = \left(9 \times 10^9 \frac{\text{Nm}^2}{\text{C}^2}\right) \frac{q_1 q_2}{r^2} \]

\[ \frac{1}{4\pi\varepsilon_0} = \left(9 \times 10^9 \frac{\text{Nm}^2}{\text{C}^2}\right) \left(1 \times 10^5 \frac{\text{dyne}}{\text{IN}}\right) \left(1 \times 10^4 \frac{\text{cm}^2}{1 \text{ m}^2}\right) \]

\[ = \left(3 \times 10^9\right)^2 \frac{\text{dyne} \cdot \text{cm}^2}{\text{C}^2} \]

Let \(3 \times 10^9\) esu = 1 C

\[ e = 1.6 \times 10^{-19} \text{C} \left(3 \times 10^9\right) \text{ esu/C} \]

\[ = 4.8 \times 10^{-10} \text{ esu} \]

\[ \vec{F}_M = q\vec{v} \times \vec{B} \quad F(\text{N}), \ q(\text{C}), \ v(\text{m/s}), \ B(\text{T}) \]

\[ = q\vec{v} \times \frac{\vec{B}}{c} \quad F(\text{dyne}), \ q(\text{esu}), \ v(\text{cm/s}), \ B(\text{G}) \]
Rutherford scattering

\( \alpha \) scattering off Gold foil

Kinetic energy of projectile.  \( E \)

Charges of projectile and target.  \( Z_e, Z'e \)

Use fine structure constant to simply calculations

\[
\frac{ZZ'e^2}{4E} = \frac{ZZ'e^2(\hbar c)}{4E(\hbar c)} = \frac{ZZ'}{4E} \frac{\alpha \hbar c}{\hbar c} = \frac{ZZ'}{4E} \frac{197 \text{ MeV}\cdot\text{fm}}{137}
\]

Cross section per unit solid angle as a function of the scattering angle, \( \theta \)

\[
\frac{d\sigma}{d\Omega}(\theta) = \left( \frac{ZZ'e^2}{4E} \right)^2 \frac{1}{\sin^4 \frac{\theta}{2}}
\]

Fine Structure Constant

\[
\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \quad (\text{SI}) = \frac{e^2}{\hbar c} \quad (\text{c.g.s}) = \frac{1}{137}
\]

\( \hbar c = 197 \text{ MeV}\cdot\text{fm} \)

1 fm = \( 1 \times 10^{-15} \text{ m} \)
Impact parameter in Rutherford scattering

Scattering from Gold nucleus

Gold nucleus
Z=79, A=197

Impact parameter

b = 7 fm

Cross sectional” area of a Gold nucleus

\begin{align*}
b &= 7 \text{ fm} \\
r &= 7 \text{ fm} = 7 \times 10^{-15} \\
A &= \pi r^2 = 154 \text{ fm}^2 = 1.54 \times 10^{-28} \text{ m}^2 \\
&= 1.54 \text{ barns} \\
1 \text{ barn} &= 1 \times 10^{-28} \text{ m}^2 = (10 \text{ fm})^2
\end{align*}

Cross sectional area of a Gold nucleus

D&F Eq. 1.32

\[ b = \frac{ZZ' e^2}{2E} \cot \frac{\theta}{2} \]

http://hyperphysics.phy-astr.gsu.edu/hphys.html
http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/crosec.html
Total Cross Section calculation

Rutherford scattering

\[
\frac{d\sigma}{d\Omega}(\theta) = \left(\frac{Z Z' e^2}{4E}\right)^2 \frac{1}{\sin^4 \frac{\theta}{2}}
\]

Trigonometry

\[
d\left(\sin \frac{\theta}{2}\right) = \frac{1}{2} \cos \frac{\theta}{2} d\theta
\]

\[
\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}
\]

\[
\sin \theta d\theta = 4 \sin \frac{\theta}{2} d\left(\sin \frac{\theta}{2}\right)
\]

See D&F deriv. of Eq. 1.32

\[
\sin^2 \frac{\theta_b}{2} = \frac{1}{1 + \frac{4b^2 E^2}{(Z Z' e^2)^2}}
\]

Total Cross Section

\[
\sigma_{\text{TOTAL}} = \int \frac{d\sigma}{d\Omega}(\theta) d\Omega \quad \text{where } d\Omega = \sin \theta d\theta d\phi
\]

\[
= \left(\frac{Z Z' e^2}{4E}\right)^2 \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \frac{1}{\sin^4 \frac{\theta}{2}}
\]

\[
= 8\pi \left(\frac{Z Z' e^2}{4E}\right)^2 \int_0^1 d\left(\sin \frac{\theta}{2}\right) \frac{1}{\sin^3 \frac{\theta}{2}}
\]

\[
= 8\pi \left(\frac{Z Z' e^2}{4E}\right)^2 \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1 + \frac{4b^2 E^2}{(Z Z' e^2)^2}}} = \pi b^2 \quad \text{Area!}
\]
Measuring cross sections in thin targets

beam particles/s $n_{\text{beam}}$

mass of target $m_T = \rho tS$

# of target moles $n_{\text{moles}} = \frac{m_T}{A}$

atomic mass $A$

# target nuclei $n_{\text{nuclei}} = n_{\text{moles}} A_0$

# target nuclei per unit area

$\frac{n_{\text{nuclei}}}{S}$

scattered particles detected per second

$n_{\text{detected}} = n_{\text{beam}} \frac{n_{\text{nuclei}}}{S} \frac{d\sigma}{d\Omega}(\theta, \phi) d\Omega_{\text{detector}}$