PHY492: Nuclear & Particle Physics

Lecture 6

Models of the Nucleus

Liquid Drop, Fermi Gas, Shell

Liquid drop model

Five terms (+ means weaker binding) in a prediction of the B.E.

- $r \sim A^{1/3}$, Binding is short ranged, depending only on nearest neighbors. This leads to a B.E. term proportional to A: $-a_1 A$.
- The surface nucleons are not surrounded by others. This leads to a term proportional to $A^{2/3}$ that weakens the B.E. : $+a_2 A^{2/3}$.
- Coulomb repulsion of the protons. This leads to a term proportional to Z^2/r that weakens the binding energy: $+a_3 (Z^2/A^{1/3})$.
- Orderly mix of p and n favors equal number of nucleons, but dilutes at big A. This leads to a B.E. term: $+a_4 (N-Z)^2/A$.
- Spin effects favor even numbers of protons or neutrons, but dilutes at big A. This leads to a term: $\pm a_5\ 1/A^{3/4}\ (Z,N)$

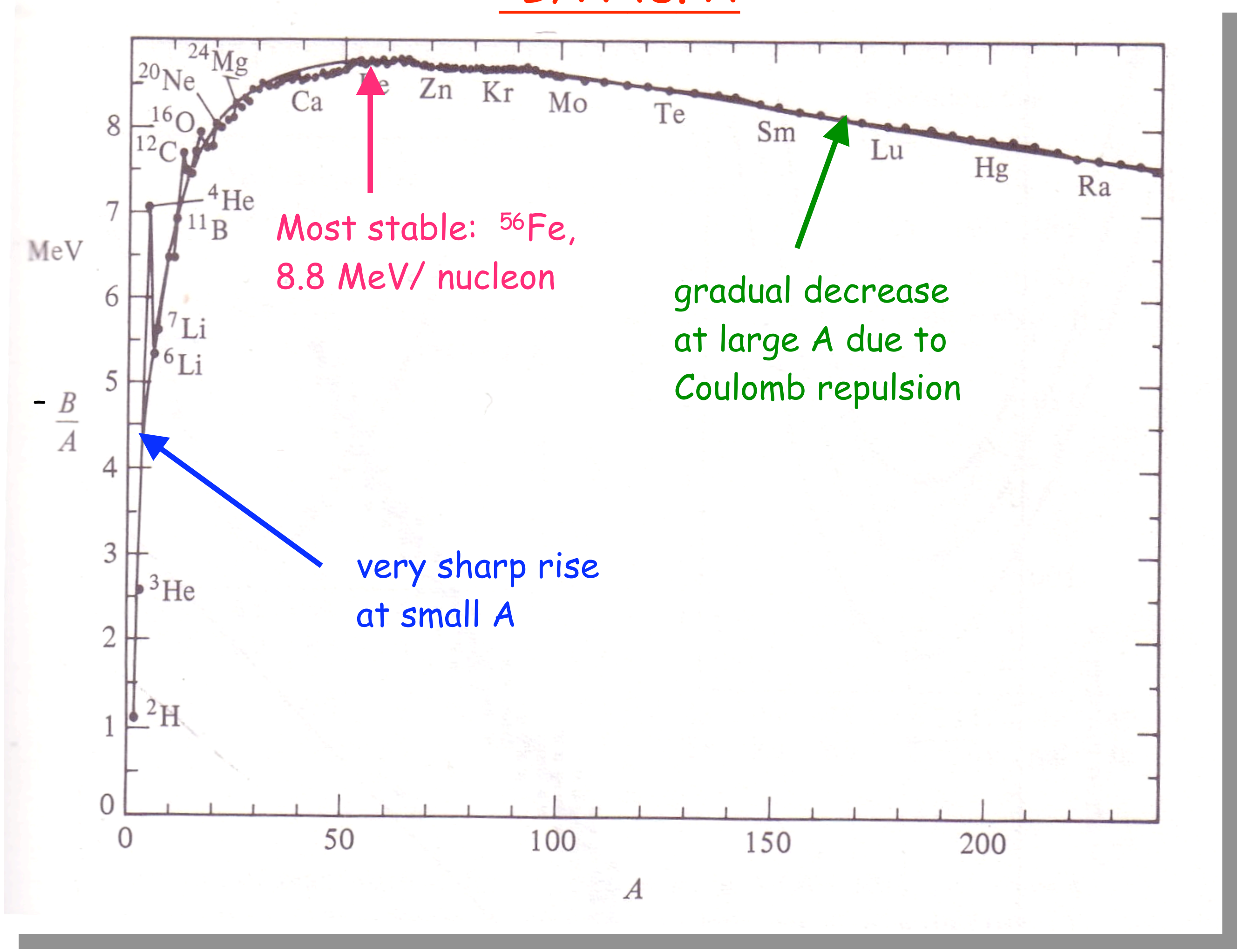
  + (odd,odd), − for (even,even), 0 for (even,odd) or (odd,even)

$$B.E.(A,Z) = -a_1 A + a_2 A^{2/3} + a_3 Z^2 A^{-1/3} + a_4 \left(2Z - A\right)^2 A^{-1} \pm a_5 A^{-3/4}$$

$$\text{in MeV: } a_1 \approx 15.6, \quad a_2 \approx 16.8, \quad a_3 \approx 0.72, \quad a_4 \approx 23.3, \quad a_5 \approx 34$$

Binding energy per nucleon

-B/A vs. A

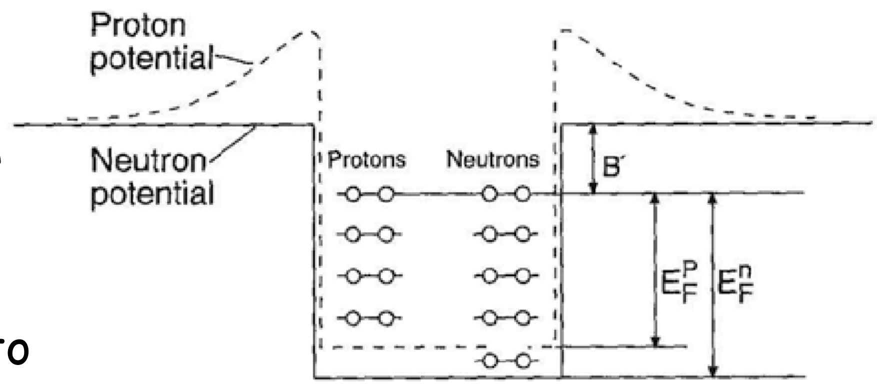


tabularized binding energies and masses - http://ie.lbl.gov/toimass.html

Fermi-gas model

Fermi-gas considerations

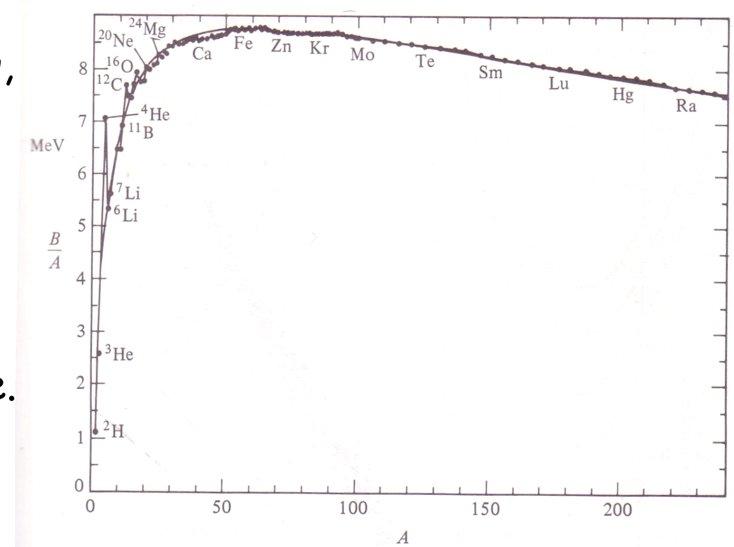
- Protons or neutrons pair-up due to spin 1/2 Fermi-Dirac statistics
- Number of nucleons that can occupy a given depth in the well is proportional to the volume and grows linearly with A.
- Well depth remains constant independent of A, at about 40 MeV.
- B' binding energy for the last nucleon remains constant independent of A, at about 8 MeV.
- E_F~(40-8)MeV. Bound nucleon momentum $(2mE_F)^{1/2}$ is constant at about 250 MeV/c



Shell model

Evidence for shell model of the nucleus

-B.E./A vs. A



- "Magic numbers" where binding is particularly strong, i.e., B.E./A is most negative.
- N = 2,8,20,28,50,82,126
  Z = 2,8,20,28,50,82
- When both N and Z are one of these, the nucleus is said to be "doubly magic" and B.E./A is most negative.
- etc.

Schrödinger equation with a "self generated" potential

In spherical coordinates with separation of variables:

$$\psi_{n\ell m_\ell}(\vec{r}) = \frac{u_{n\ell}(r)}{r} Y_{\ell m_\ell}(\theta,\phi)$$

Radial equation:
$$\left(\frac{d^2}{dr^2} + \frac{2m}{\hbar^2}\left(E_{n\ell} - V(r) - \frac{\hbar^2 \ell(\ell+1)}{2mr^2}\right)\right) u_{n\ell}(r) = 0$$

Shell model potentials

Infinite square well

$$V(r) = \begin{cases} \infty & r \geq R \\ 0 & \text{otherwise} \end{cases}$$

Solution of radial equation:

$$u_{n\ell}(r) = j_\ell(k_{n\ell} r); \quad k_{n\ell} = \left(\frac{2mE_{n\ell}}{\hbar^2}\right)$$

spherical Bessel functions

Boundary condition at edge of infinite well:

$$j_\ell(k_{n\ell} R) = 0 \qquad E_{n\ell} = \frac{k_{n\ell}\hbar^2}{2m}$$

$$\ell = 0,1,2,3,\ldots; \quad \text{and} \quad n = 1,2,3,\ldots \quad \text{for any } \ell$$

For each ℓ, there are $2\ell+1$ states (different m_ℓ) and each shell can contain $2(2\ell+1)$ protons or neutrons

Gives: 2, 8, 18, 32, 50 ... as number in filled shells for n=1).

Shouldn't have 18,32 and is missing 20, 82 and 126.

Harmonic Oscillator

energy eigenvalues: $E_{n\ell} = \hbar\omega\left(2n + \ell - \tfrac{1}{2}\right)$

Gives: 2, 8,20, 40, 70 ... as number in filled shells

Shouldn't have 40,70 and is missing 50, 82 and 126

Shell model potentials

Spin-Orbit

$$V = V(r) - f(r) L \cdot S$$

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$$\langle \psi | \vec{L} \cdot \vec{S} | \psi \rangle = \tfrac{\hbar^2}{2}\left(j(j+1) - \ell(\ell+1) - s(s+1) \right); \quad s = \tfrac{1}{2}$$

$$= \begin{cases} \tfrac{\hbar^2}{2}\ell & \text{for } j = \ell + \tfrac{1}{2} \\ \tfrac{\hbar^2}{2}(\ell+1) & \text{for } j = \ell - \tfrac{1}{2} \end{cases}$$

In addition to energies derived from V(r) there are energy corrections:

$$\Delta = \hbar^2 \left(\ell + \tfrac{1}{2} \right) \int d^3 r \left| \psi_{n\ell}(\vec{r}) \right|^2 f(r)$$

Energy corrections can promote a state n with $\ell > 0$, above the n + 1, $\ell = 0$ state.

Spectroscopic Notation: $nX_j, j^{\pm}; \quad X = S, P, D, F, G, H \ldots \quad \text{for } \ell = 0,1,2,3,4,5\ldots$

Total angular momentum & parity: j^{+} for $\ell = 0,2,4,\ldots$ $\quad j^{-}$ for $\ell = 1,3,5,\ldots$

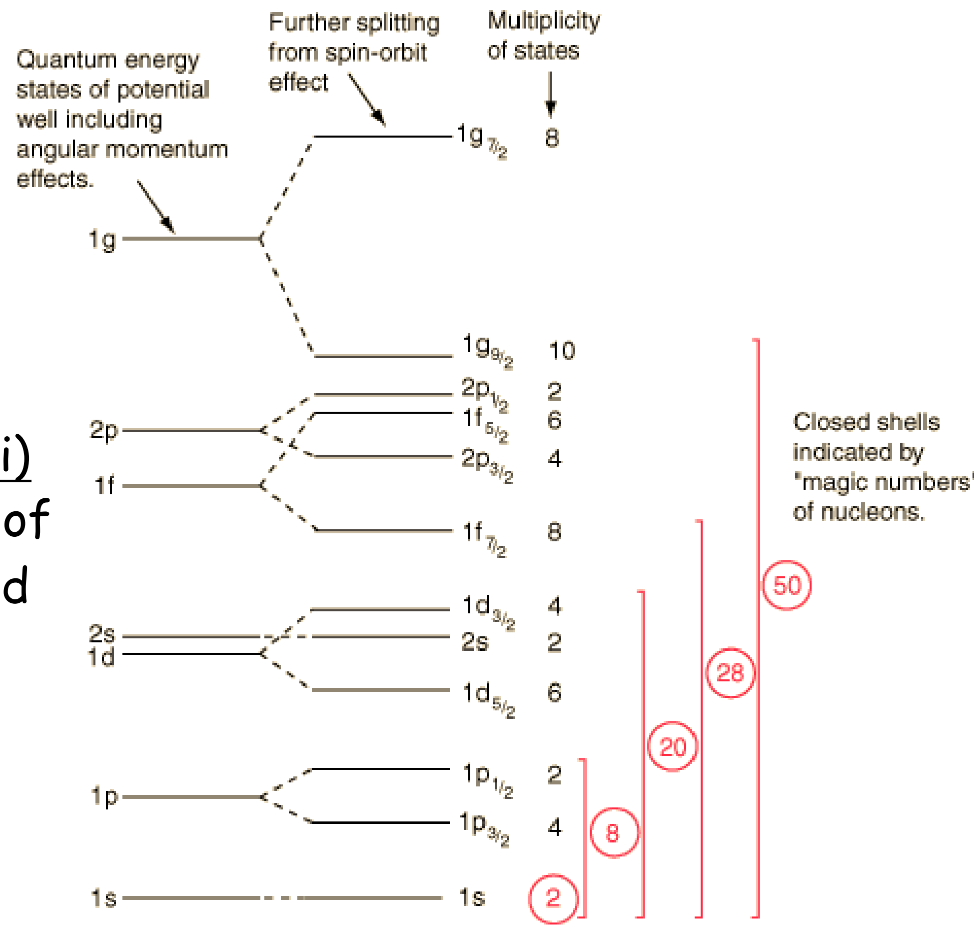
Gives energy clusters with 2,6,12,8,22,8 nucleons
Results in "magic numbers": **2**, 2+6=**8**, 8+12=**20**, 20+8=**28**, 28+22=**50**, ...

Consequences of Spin-Orbit Shell Model

Use energy level diagram on page 72 of Das and Ferbel.

Nuclear angular momentum, j and parity

Fill shells from the bottom up, independently with protons and neutrons, to determine the total angular momentum j , and the parity (±, based on ℓ), of nucleus based on last unpaired nucleons.



Nuclear magnetic moment (light nuclei)

Use anomalous magnetic moments of unpaired proton or neutron and add any orbital angular momentum ℓ,

$$\mu_\ell = \frac{e\hbar}{2m}\ell$$

to obtain the nuclear magnetic moment.

Measured values found on www.nndc.bnl.gov

Example from problem 3.5

Spin (j) and parity for ground states of $^{23}\text{Na}^{11}$, $^{35}\text{Cl}^{17}$, $^{41}\text{Ca}^{20}$

$^{23}\text{Na}^{11}$ 12 neutrons — all neutrons pair up
11 protons — 10 protons pair up

$(1S_{1/2})^2,(1P_{3/2})^4,(1P_{1/2})^2,(1D_{5/2})^2$

last 1 proton in $1D_{5/2}$

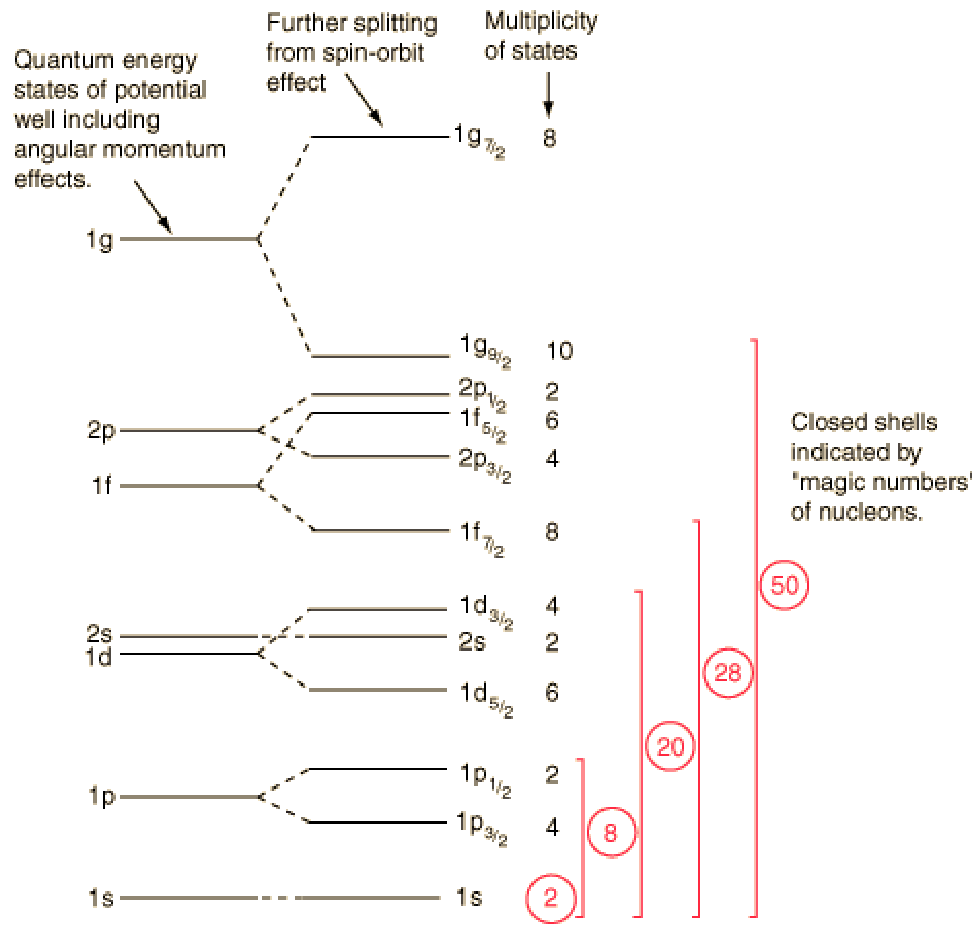
Predict: $j^P = \frac{5}{2}^+$

Actual: $j^P = \frac{3}{2}^+$

Last proton is in $1D_{3/2}$

Must be lower energy than

$1D_{5/2}$ (or $2S_{1/2}$)



Magnetic Moments

protons $\mu = (2.79 + \ell)\mu_N$

neutrons $\mu = -1.92\mu_N$

Predict: $\mu = (2.79 + 2)\mu_N$ Actual: $\mu = 2.22\mu_N$

Problem 2.11 hints and fix

Hint: Expansion

$$\text{expand } e^{\frac{i}{\hbar}\vec{q}\cdot\vec{r}} = 1 + ikr\cos\theta - \tfrac{1}{2}k^2r^2\cos^2\theta; \qquad \vec{k} = \frac{1}{\hbar}\vec{q};$$

Volume element

$$d^3r = r^2 dr d\left(\cos\theta\right) d\phi$$

Fix: Use this form for Gaussian r distribution

$$\rho\left(\vec{r}\right) = \frac{1}{R^3}\frac{1}{\left(2\pi\right)^{3/2}} e^{-\frac{r^2}{2R^2}}$$

R is standard deviation

Nuclear Radiation

Types of radiation from a nucleus

- Alpha (α)
  - Helium nucleus,
  - T = 0 - 10 MeV
  - stopped by a few paper sheets
- Beta ($\beta^{\pm}$)
  - electrons (–), positrons (+)
  - T = 0 - 3.5 MeV
  - stopped by 2 cm of plastic
- Gamma (γ)
  - photons
  - E = 0 - 5 MeV
  - most stopped by a 5 mm of lead.
- Electron capture (EC)
  - only neutrino radiated
  - effect on the nucleus is the same as a β^{+} decay

Gamma radiation

- Typical half-life for radiation of a photon, is a few ps (10^{-12}s). None are left after 1 ns (10^{-9}s)
- Rare "meta-stable" states exist that have longer gamma lifetimes.
- All gamma emitting nuclei come from a preceding α,β,EC, or fission
- Most gamma energies are quantized into spectral "lines".
- Spectral lines reflect the nuclear level structure