Lecture 6
Models of the Nucleus
Liquid Drop, Fermi Gas, Shell
Liquid drop model

Five terms (+ means weaker binding) in a prediction of the B.E.

- \( r \sim A^{1/3} \), Binding is short ranged, depending only on nearest neighbors. This leads to a B.E. term proportional to \( A \): \(-a_1 A\).

- The surface nucleons are not surrounded by others. This leads to a term proportional to \( A^{2/3} \) that weakens the B.E.: \(+a_2 A^{2/3}\).

- Coulomb repulsion of the protons. This leads to a term proportional to \( Z^2/r \) that weakens the binding energy: \(+a_3 (Z^2/A^{1/3})\).

- Orderly mix of p and n favors equal number of nucleons, but dilutes at big A. This leads to a B.E. term: \(+a_4 (N-Z)^2/A\).

- Spin effects favor even numbers of protons or neutrons, but dilutes at big A. This leads to a term: \(\pm a_5 1/A^{3/4} \) \((Z,N)\)
  + (odd,odd), \(-\) for (even,even), 0 for (even,odd) or (odd,even)

\[
B.E.(A,Z) = -a_1 A + a_2 A^{2/3} + a_3 Z^2 A^{-1/3} + a_4 (2Z - A)^2 A^{-1} \pm a_5 A^{-3/4}
\]

in MeV: \( a_1 \approx 15.6, \quad a_2 \approx 16.8, \quad a_3 \approx 0.72, \quad a_4 \approx 23.3, \quad a_5 \approx 34 \)
Binding energy per nucleon

-B/A vs. A

Most stable: $^{56}$Fe, 8.8 MeV/nucleon

gradual decrease at large A due to Coulomb repulsion

very sharp rise at small A

tabularized binding energies and masses - http://ie.lbl.gov/toimass.html
Fermi-gas model

Fermi-gas considerations

- Protons or neutrons pair-up due to spin 1/2 Fermi-Dirac statistics
- Number of nucleons that can occupy a given depth in the well is proportional to the volume and grows linearly with \( A \).
- Well depth remains constant independent of \( A \), at about 40 MeV.
- \( B' \) binding energy for the last nucleon remains constant independent of \( A \), at about 8 MeV.
- \( E_F \approx (40-8)\) MeV. Bound nucleon momentum \( (2mE_F)^{1/2} \) is constant at about 250 MeV/c
Evidence for shell model of the nucleus

- “Magic numbers” where binding is particularly strong, i.e., B.E./A is most negative.
- N = 2, 8, 20, 28, 50, 82, 126
  Z = 2, 8, 20, 28, 50, 82
- When both N and Z are one of these, the nucleus is said to be “doubly magic” and B.E./A is most negative.
- etc.

In spherical coordinates with separation of variables:

Schrödinger equation with a “self generated” potential

Radial equation:

\[
\left( \frac{d^2}{dr^2} + \frac{2m}{\hbar^2} \left( E_{n\ell} - V(r) - \frac{\hbar^2 \ell (\ell + 1)}{2mr^2} \right) \right) u_{n\ell}(r) = 0
\]
Shell model potentials

Infinite square well

\[ V(r) = \begin{cases} \infty & r \geq R \\ 0 & \text{otherwise} \end{cases} \]

Solution of radial equation:

\[ u_{n\ell}(r) = j_\ell(k_{n\ell} r); \quad k_{n\ell} = \left( \frac{2mE_{n\ell}}{\hbar^2} \right) \]

Boundary condition at edge of infinite well:

\[ j_\ell(k_{n\ell} R) = 0 \]

\[ E_{n\ell} = \frac{k_{n\ell} \hbar^2}{2m} \]

For each \( \ell \), there are \( 2\ell + 1 \) states (different \( m_\ell \)) and each shell can contain \( 2(2\ell + 1) \) protons or neutrons.

Gives: 2, 8, 18, 32, 50 ... as number in filled shells for \( n=1 \).

Shouldn’t have 18,32 and is missing 20, 82 and 126.

Harmonic Oscillator

Energy eigenvalues:

\[ E_{n\ell} = \hbar \omega \left( 2n + \ell - \frac{1}{2} \right) \]

Gives: 2, 8,20, 40, 70 ... as number in filled shells

Shouldn’t have 40,70 and is missing 50, 82 and 126.
Shell model potentials

Spin-Orbit

\[ V = V(r) - f(r) \mathbf{L} \cdot \mathbf{S} \]

Maria Goeppert Mayer & Hans Jensen

\[ \langle \psi \mid \mathbf{L} \cdot \mathbf{S} \mid \psi \rangle = \frac{\hbar^2}{2} \left( j(j+1) - \ell(\ell+1) - s(s+1) \right) ; \quad s = \frac{1}{2} \]

\[ = \begin{cases} \frac{\hbar^2}{2} \ell & \text{for } j = \ell + \frac{1}{2} \\ \frac{\hbar^2}{2}(\ell + 1) & \text{for } j = \ell - \frac{1}{2} \end{cases} \]

In addition to energies derived from \( V(r) \) there are energy corrections:

\[ \Delta = \hbar^2 \left( \ell + \frac{1}{2} \right) \int d^3r \left| \psi_{n\ell}(\mathbf{r}) \right|^2 f(r) \]

Energy corrections can promote a state \( n \) with \( \ell > 0 \), above the \( n + 1, \ell = 0 \) state.

Spectroscopic Notation:

\[ nX_j, j^\pm; \quad X = S, P, D, F, G, H \ldots \quad \text{for } \ell = 0, 1, 2, 3, 4, 5 \ldots \]

Total angular momentum & parity:

\[ j^+ \quad \text{for } \ell = 0, 2, 4, \ldots \quad j^- \quad \text{for } \ell = 1, 3, 5, \ldots \]

Gives energy clusters with 2, 6, 12, 8, 22, 8 nucleons

Results in “magic numbers”: \[ 2, \ 2+6=8, \ 8+12=20, \ 20+8=28, \ 28+22=50, \ldots \]
Consequences of Spin-Orbit Shell Model

Use energy level diagram on page 72 of Das and Ferbel.

Nuclear angular momentum, j and parity

Fill shells from the bottom up, independently with protons and neutrons, to determine the total angular momentum \( j \), and the parity \( (\pm, \text{based on } \ell) \), of nucleus based on last unpaired nucleons.

Nuclear magnetic moment (light nuclei)

Use anomalous magnetic moments of unpaired proton or neutron and add any orbital angular momentum \( \ell \),

\[
\mu_\ell = \frac{e\hbar}{2m} \ell
\]

to obtain the nuclear magnetic moment.

Measured values found on www.nndc.bnl.gov
Example from problem 3.5

**Spin (j) and parity for ground states of**

12 neutrons  all neutrons pair up
11 protons  10 protons pair up

\[ (1S_{1/2})^2, (1P_{3/2})^4, (1P_{1/2})^2, (1D_{5/2})^2 \]

last 1 proton in \( 1D_{5/2} \)

**Predict:** \( j^P = \frac{5}{2}^+ \)

**Actual:** \( j^P = \frac{3}{2}^+ \)

Last proton is in \( 1D_{3/2} \)

**Must be lower energy than**

\( 1D_{5/2} \) (or \( 2S_{1/2} \))

**Magnetic Moments**

protons \( \mu = (2.79 + \ell)\mu_N \)
neutrons \( \mu = -1.92\mu_N \)

**Predict:** \( \mu = (2.79 + 2)\mu_N \)  
**Actual:** \( \mu = 2.22\mu_N \)
Problem 2.11 hints and fix

**Hint: Expansion**

\[ \exp(\frac{i}{\hbar}\vec{q} \cdot \vec{r}) = 1 + ikr \cos \theta - \frac{1}{2} k^2 r^2 \cos^2 \theta; \quad \vec{k} = \frac{1}{\hbar} \vec{q}; \]

**Volume element**

\[ d^3r = r^2 dr d(\cos \theta) d\phi \]

**Fix: Use this form for Gaussian r distribution**

\[ \rho(\vec{r}) = \frac{1}{R^3} \frac{1}{(2\pi)^{3/2}} e^{-\frac{r^2}{2R^2}} \]

R is standard deviation
Nuclear Radiation

Types of radiation from a nucleus

- **Alpha (α)**
  - Helium nucleus,
  - \( T = 0 - 10 \text{ MeV} \)
  - stopped by a few paper sheets

- **Beta (β⁺)**
  - electrons (-), positrons (+)
  - \( T = 0 - 3.5 \text{ MeV} \)
  - stopped by 2 cm of plastic

- **Gamma (γ)**
  - photons
  - \( E = 0 - 5 \text{ MeV} \)
  - most stopped by a 5 mm of lead.

- **Electron capture (EC)**
  - only neutrino radiated
  - effect on the nucleus is the same as a \( \beta^+ \) decay

**Gamma radiation**

- Typical half-life for radiation of a photon, is a few ps \( (10^{-12}s) \). **None** are left after 1 ns \( (10^{-9}s) \)
- Rare “meta-stable” states exist that have longer gamma lifetimes.
- All gamma emitting nuclei come from a preceding \( \alpha, \beta, \text{EC}, \) or fission
- Most gamma energies are quantized into spectral “lines”.
- Spectral lines reflect the nuclear level structure