

## 1. Complex Numbers

While the values of physical observables must be real numbers, quantum mechanical amplitudes are, in general, complex numbers. Consequently, some facility with the properties of complex numbers is needed in any study of quantum mechanics.

A complex number  $z$  is usually represented as  $z = x + iy$ , where  $x$  and  $y$  are real numbers and the symbol  $i$  satisfies  $i^2 = -1$ .  $i$  can be viewed as  $\sqrt{-1}$ . Apart from the introduction of  $i$ , the algebraic properties of complex numbers are the same as those of real numbers. For example,

$$z^2 = (x + iy)^2 = x^2 + 2x(iy) + (iy)^2 = x^2 - y^2 + 2ixy.$$

Notice that  $z^2$  is another complex number and that, unlike the square of a real number, it is not positive. There is a positive number naturally associated with a complex number  $z$ . It is best constructed by defining the *complex conjugate*,  $z^*$ , of  $z$ . To get the complex conjugate of  $z$ , one simply changes  $i$  to  $-i$ , i.e.

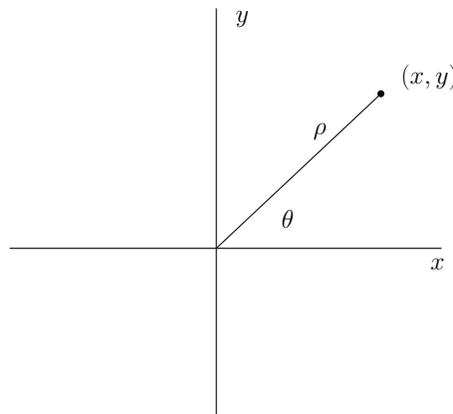
$$z^* = (x + iy)^* = x - iy.$$

With this definition, the product  $z^*z$  is

$$|z|^2 \equiv z^*z = (x - iy)(x + iy) = x^2 + y^2 \geq 0.$$

$|z| = \sqrt{x^2 + y^2}$  is called the absolute value or *modulus* of  $z$ .

Complex numbers reside in a plane, called the complex plane, whose  $x$ -axis is the *real* axis and whose  $y$ -axis is the *imaginary* axis. The ‘point’  $(x, y) \leftrightarrow x + iy$  can also be specified by the



polar coordinates  $\rho, \theta$  as

$$z = \rho(\cos \theta + i \sin \theta), \quad \rho = \sqrt{x^2 + y^2}, \quad \tan \theta = \frac{y}{x}.$$

Typically, the polar form is rewritten using Euler's formula

$$\cos \theta + i \sin \theta = e^{i\theta},$$

to obtain

$$z = x + iy = \rho e^{i\theta}.$$

In the latter form,  $\rho$  is just  $|z|$ , the modulus of  $z$  and  $\theta$  is called the *phase* of  $z$ .

### Exercises

1. Find the product of the complex numbers  $2 + 3i$  and  $1 + 4i$ .
2. Determine the reciprocal of  $z = x + iy$  in terms of  $x$  and  $y$ . Hint: If the reciprocal is  $\zeta = \chi + i\psi$ , then  $\zeta z = 1$ . Use this to determine  $\chi$  and  $\psi$ .
3. Show that  $|e^{i\theta}| = 1$  using Euler's formula.
4. Express  $z = 2 + 2i$  in polar form.
5. If  $\alpha$  and  $\beta$  are real, use Euler's formula to show that

$$e^{i\alpha} e^{i\beta} = e^{i(\alpha+\beta)}.$$

6. Find the Cartesian form of  $z = 4e^{i\pi/6}$ .
7. What can you conclude about  $z$  if  $z = z^*$ ?

## 2. Differential Equations

In quantum mechanics, like classical mechanics, the dynamics of particles are determined by a differential equation. The quantum mechanical differential equation is called Schrödinger's equation. In one dimension, it is a second order differential equation for the wave function  $\psi(x)$ . As such, its solution will, in general, involve finding two independent functions which satisfy the differential equation and imposing certain 'boundary' conditions on them dictated by the physical circumstances. For many interesting applications, the Schrödinger equation for  $\psi(x)$  can be reduced to the form

$$\frac{d^2\psi(x)}{dx^2} \pm k^2\psi(x) = 0,$$

with  $k$  being a real number. The form of the solution is determined by the sign of the  $k^2$  term. If it is  $+k^2$ , then the equation has the same form as the classical harmonic oscillator, and the general solution is

$$\psi(x) = A \cos(kx) + B \sin(kx),$$

where  $A$  and  $B$  are independent of  $x$  and determined by the boundary conditions. For the  $-k^2$  case, the general solution is

$$\psi(x) = A e^{kx} + B e^{-kx}.$$

In quantum mechanical applications, the boundary conditions usually restrict not only the form of the solution (oscillating or exponential), but also the values of  $k$  that are permissible. The validity of these solutions and the use of boundary conditions are illustrated in the following exercises.

### Exercises

1. Show by direct differentiation that the form of  $\psi(x)$  for the  $-k^2$  case is a solution to the differential equation.
2. Show that the  $+k^2$  oscillatory solution can also be written

$$\psi(x) = Ce^{ikx} + De^{-ikx}.$$

3. For the oscillatory ( $+k^2$ ) solution, suppose  $x$  is restricted to  $0 \leq x \leq a$ . By requiring  $\psi(0) = 0$  and  $\psi(a) = 0$ , what are the restrictions on  $A$ ,  $B$ , and  $k$ ?
4. In some quantum mechanics problems, there are several regions with different values of  $k$ . Suppose for  $0 \leq x \leq a$ , the Schrödinger equation is

$$\frac{d^2\psi(x)}{dx^2} + \kappa^2\psi(x) = 0,$$

and, for  $a \leq x \leq \infty$ , it is

$$\frac{d^2\psi(x)}{dx^2} - k^2\psi(x) = 0.$$

Determine the form of  $\psi(x)$  in the two regions if  $\psi(0) = 0$  and as  $x \rightarrow \infty$ ,  $\psi(x) \rightarrow 0$ .

### 3. Matrices

For quantum systems with a finite number of degrees of freedom, the equation of motion can be written as a relation involving finite dimensional square matrices. We will develop this notion in some detail in class, but it is useful to recall some simple properties of matrices. A square matrix  $A$  is an  $n \times n$  array usually written as

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}.$$

In quantum mechanics, the notion of the *hermiticity* of a matrix is very important. Its definition involves complex conjugation, discussed above, and the transpose,  $A^T$ , of the matrix  $A$ .  $A^T$  is the matrix obtained by interchanging the row and columns of  $A$ , and the hermitian conjugate of  $A$ ,  $A^\dagger$ , is the complex conjugate of this matrix. Explicitly,  $A^\dagger$  is

$$A^\dagger = \begin{pmatrix} a_{11}^* & a_{21}^* & a_{31}^* & \cdots & a_{n1}^* \\ a_{12}^* & a_{22}^* & a_{32}^* & \cdots & a_{n2}^* \\ a_{13}^* & a_{23}^* & a_{33}^* & \cdots & a_{n3}^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n}^* & a_{2n}^* & a_{3n}^* & \cdots & a_{nn}^* \end{pmatrix}.$$

$A^T$  is just this matrix with the \*'s omitted. If  $A^\dagger = A$ , the matrix  $A$  is said to be hermitian and if  $A^T = A$ ,  $A$  is said to be symmetric.

Matrix multiplication is defined by the 'row  $\times$  column' rule: If  $A$  and  $B$  are two  $n \times n$  matrices, then the  $i, j^{\text{th}}$  element  $(AB)_{ij}$  of their product is

$$(AB)_{ij} = \sum_{k=1}^n a_{ik} b_{kj}.$$

In general, matrix multiplication is not commutative, i.e. usually  $AB \neq BA$ . The difference  $AB - BA$  is called the commutator of  $A$  and  $B$  and is usually denoted  $[A, B]$ . Under certain conditions a matrix  $A$  may have an *inverse*,  $A^{-1}$ , which satisfies  $AA^{-1} = \mathbf{1}$ . Here,  $\mathbf{1}$  denotes the unit matrix,

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

### Exercises

1. Which of the following matrices is hermitian

$$A = \begin{pmatrix} 2 & i \\ i & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad C = \begin{pmatrix} 3 & 1+2i & 0 \\ 1-2i & 4 & 2 \\ 0 & 2 & 0 \end{pmatrix} \quad D = \begin{pmatrix} i & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

2. Compute  $CD$ .
3. Compute  $[A, B]$ .
4. Compute  $B^2$ . What can you conclude about  $B^{-1}$ ?
5. Show that the unit matrix commutes with any matrix.
6. What can you say about the diagonal elements of an hermitian matrix?