PHYSICS 471, FALL 2008

POLARIZATION OF LIGHT

(These notes are based on the presentation in Lectures on Quantum Mechanics by Gordon Baym, as well as lecture notes by Prof. Roger Tobin of Tufts University.)

Consider an electromagnetic plane wave traveling in the +z direction. We can write the electric field vector components for such a wave as follows:

$$E_x(x,t) = \mathrm{Re}\{E_x \exp(i(kx - \omega t))\} \qquad E_y(x,t) = \mathrm{Re}\{E_y \exp(i(kx - \omega t))\}$$

Comments:

i) If $E_y = 0$, then the wave is linearly polarized in the x direction.

ii) If $E_x = 0$, then the wave is linearly polarized in the y direction.

iii)If $E_y = E_x$, then the wave is linearly polarized along the 45° direction.

iv) If $E_y = e^{i\pi/2}E_x = iE_x$, then the y-component lags the x-component by 90°, and the wave is right circularly polarized. Similarly, if $E_y = -iE_x$, then the wave is left circularly polarized.

How large is the electric field for a single photon? That question has a definite answer only if our photon is inside a box of volume V. The energy density in an electromagnetic wave is equal to: $u = \frac{|\vec{E}|^2}{2\varepsilon_0}$. If all the energy is in a single plane wave mode, then $uV = N\hbar\omega$ where N is the number of photons. For a single photon, we have

$$uV = \frac{|\vec{E}|^2}{2\varepsilon_0}V = \hbar\omega \qquad \text{or} \qquad |\vec{E}| = \sqrt{\frac{2\varepsilon_0\hbar\omega}{V}}.$$

For a single photon, we can define the following normalized electric field components:

$$\psi_x = \sqrt{\frac{V}{2\varepsilon_0\hbar\omega}}E_x \qquad \psi_y = \sqrt{\frac{V}{2\varepsilon_0\hbar\omega}}E_y \qquad \text{with} \qquad \psi_x^2 + \psi_y^2 = 1$$

Now let's describe the photon polarization using the language of quantum mechanics. We'll use the "ket" $|\psi\rangle$ to describe the polarization state, with ψ_x and ψ_y the components of $|\psi\rangle$ in the basis of linear polarization along the x and y axes. We can write $|\psi\rangle$ as a linear combination of basis kets, or associate it with a column vector:

$$|\psi\rangle = \psi_x |x\rangle + \psi_y |y\rangle \rightarrow \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix}$$

where the basis kets are: $|x\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|y\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

They form an orthonormal basis:

$$\langle x|x\rangle = \langle y|y\rangle = 1$$
$$\langle x|y\rangle = \langle y|x\rangle = 0$$

Consider the basis rotated by an angle θ in the x-y plane with respect to our original basis. The two new basis kets are then:

$$|\theta\rangle = \cos(\theta)|x\rangle + \sin(\theta)|y\rangle$$

$$\left|\theta + \frac{\pi}{2}\right\rangle = -\sin\theta|x\rangle + \cos\theta|y\rangle$$

It is easy to verify that these two kets also form an orthonormal basis:

$$\langle\theta|\theta\rangle = \left\langle\theta + \frac{\pi}{2}\middle|\theta + \frac{\pi}{2}\right\rangle = 1$$
$$\left\langle\theta\middle|\theta + \frac{\pi}{2}\right\rangle = \left\langle\theta + \frac{\pi}{2}\middle|\theta\right\rangle = 0 \quad .$$

A third useful basis consists of circularly-polarized light, $|R\rangle$ and $|L\rangle$ for right and left circularly polarized light, respectively. They are defined this way:

$$|R\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle + i|y\rangle\right) = \frac{e^{i\theta}}{\sqrt{2}}\left(|\theta\rangle + i\left|\theta + \frac{\pi}{2}\right\rangle\right)$$
$$|L\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle - i|y\rangle\right) = \frac{e^{-i\theta}}{\sqrt{2}}\left(|\theta\rangle - i\left|\theta + \frac{\pi}{2}\right\rangle\right)$$

Classically, the *R* and *L* states represent light for which the magnitude of the electric field is constant, but its direction rotates around the direction of propagation. Quantum mechanically, the *R* and *L* states are of interest because they are eigenstates of the photon's spin angular momentum along the direction of propagation.

Linear Polarizers

In class, we will do some simple experiments with linear polarizers. A linear polarizer only allows light to pass through that is linearly polarized in a particular direction. In quantum mechanics, a polarizer is represented by a projection operator. (I will use the symbol $\hat{P}$ to mean a projection operator in these notes. It has nothing to do with momentum.) We can write the operator either in terms of bras and kets, or as a matrix (with the implicit understanding that we are in the original basis discussed above):

$$\hat{P}_x = |x\rangle\langle x| \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Similarly, a linear polarizer oriented along the y-direction is represented by the operator:

$$\hat{P}_y = |y\rangle\langle y| \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Notice that the sum of these two projection operators is the identity operator. That is just the famous Completeness Relation I have mentioned many times in class:

$$\hat{I} = |x\rangle\langle x| + |y\rangle\langle y| = \sum_n |n\rangle\langle n| \qquad \text{where } n = x, y$$

A linear polarizer directed along an arbitrary angle θ with respect to the x-axis is represented by the operator:

$$\hat{P}_\theta = |\theta\rangle\langle\theta| = \left(\cos(\theta)|x\rangle + \sin(\theta)|y\rangle\right)\left(\cos(\theta)\langle x| + \sin(\theta)\langle y|\right) \rightarrow \begin{pmatrix} \cos^2(\theta) & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \cos^2(\theta) \end{pmatrix}$$

Consider the following experiment: We start with light of unknown polarization. First we pass it through an x-polarizer, then we pass the outgoing light through a y-polarizer. What do we get? Let's say that the initial light is in the state:

$$|\psi_{initial}\rangle = a|x\rangle + b|y\rangle \rightarrow \begin{pmatrix} a \\ b \end{pmatrix}$$

After the first polarizer, we get:

$$|\psi_{x-out}\rangle = P_x|\psi_{initial}\rangle = P_x\left(a|x\rangle + b|y\rangle\right) = \left(|x\rangle\langle x|\right)\left(a|x\rangle + b|y\rangle\right) = a|x\rangle$$

(We are not insisting that our states be normalized now, because we will keep track of the light intensity by taking the modulus squared of our final vector.)

In matrix notation, the same calculation is written as:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$$

After going through the second polarizer, we get:

$$\left|\psi_{y-out}\right\rangle = P_y\left|\psi_{x-out}\right\rangle = P_y\left(a|x\rangle\right) = \left(|y\rangle\langle y|\right)\left(a|x\rangle\right) = a|y\rangle\langle y|x\rangle = |0\rangle$$

Or in matrix form:

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

In other words, no light passes through the second polarizer!

Now, what happens if we insert a third linear polarizer, oriented at an angle θ, <u>between</u> the x and y polarizers? After passing through the x-polarizer, we get $\left|\psi_{x-out}\right\rangle$ again.

After passing through the second polarizer, we get:

$$\left|\psi_{\theta-out}\right\rangle = P_\theta\left|\psi_{x-out}\right\rangle == \left(|\theta\rangle\langle\theta|\right)\left(a|x\rangle\right) = a|\theta\rangle\langle\theta|x\rangle = a\cos(\theta)|\theta\rangle$$

Or in matrix notation:

$$\begin{pmatrix} \cos^2(\theta) & \cos(\theta)\sin(\theta) \\ \cos(\theta)\sin(\theta) & \cos^2(\theta) \end{pmatrix}\begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} a\cos^2(\theta) \\ a\cos(\theta)\sin(\theta) \end{pmatrix} = a\cos(\theta)\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

After passing through the third polarizer, we get the final result:

$$\left|\psi_{y-out}\right\rangle = P_y\left|\psi_{\theta-out}\right\rangle == \left(|y\rangle\langle y|\right)\left(a\cos(\theta)|\theta\rangle\right) = a\cos(\theta)|y\rangle\langle y|\theta\rangle = a\cos(\theta)\sin(\theta)|y\rangle$$

In matrix notation:

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} a\cos(\theta)\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} = a\cos(\theta)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} = a\cos(\theta)\sin(\theta)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The intensity of the final light is

$$Intensity = \left\langle\psi_{y-out}\middle|\psi_{y-out}\right\rangle = |a|^2\cos^2(\theta)\sin^2(\theta)$$

So we got <u>more</u> light through by putting an additional polarizing filter into the apparatus!

It turns out that all of these results about light can be obtained from classical physics, but that won't be true when we look at other quantum-mechanical problems, such as electron spin.