Chapter 9

Dynamics of a System of Particles

Center of Mass

Consider \( n \) particles labelled by \( \alpha = 1, 2, \ldots, n \). The total mass of these particles is

\[
M = \sum_{\alpha} m_{\alpha}
\]

Suppose that the position of each particle relative to the origin is \( \vec{r}_{\alpha} \).

Then the position of the CM of the system is

\[
\vec{R} = \frac{1}{M} \sum_{\alpha} m_{\alpha} \vec{r}_{\alpha}
\]
For a continuous distribution of mass (e.g., a solid body) the summation is replaced by an integral

\[ \mathbf{R} = \frac{1}{M} \int \mathbf{r} \, dm \]

(There is a good example (9.1) of how to find the CM of a solid body in the text book.)

What's so special about the CM?

1. CM = center of gravity
   If you hang an object from any point, the CM is directly below that point. (Or you can balance an object if the CM is directly above or below.)

2. But, more important, in the case of the motion of complicated systems of many particles, the motion of the CM is often very simple.
Consider a system of \( n \) particles:

Linear momentum of the system, \( \vec{p} \) is

\[
\vec{p} = \sum_{\alpha} \vec{p}_\alpha = \sum_{\alpha} m_\alpha \vec{r}_\alpha
\]

\[
= \frac{d}{dt} \left( \sum_{\alpha} m_\alpha \vec{r}_\alpha \right) = \frac{d}{dt} \left( M \vec{R} \right)
\]

assuming that the mass of the system doesn't change

\[
\therefore \vec{p} = M \vec{R}
\]

so total momentum = total mass \( \times \) velocity of the CM

Now how does it behave under external forces?

The force on the \( \alpha \)th particle can be written

\[
\vec{F}_\alpha = \vec{F}_\alpha^{\text{ext}} + \vec{F}_\alpha^{\text{int}}
\]

↑

external force

↑

internal force due to the \( n-1 \) other particles
where \[ \sum_{\alpha} \vec{F}_{\alpha} = \sum_{\beta} \vec{F}_{\alpha \beta} \]

representing the force on the \( \alpha \)th particle due to
the \( \beta \)th particle.

Each particle obeys Newton’s 2nd Law
\[ \vec{F}_{\alpha} = \vec{F}_{\alpha \text{ext}} + \sum_{\beta} \vec{F}_{\alpha \beta} = \frac{d}{dt} (m_{\alpha} \vec{r}_{\alpha}) \]

Summing over all particles \( \Rightarrow \)
\[ \sum_{\alpha} \vec{F}_{\alpha \text{ext}} + \sum_{\alpha} \sum_{\beta} \vec{F}_{\alpha \beta} = \frac{d}{dt} \left( \sum_{\alpha} m_{\alpha} \vec{r}_{\alpha} \right) \]

\( \uparrow \)
\[ \Rightarrow \quad \vec{F}_{\text{ext}} = \frac{d}{dt} (M \vec{R}) \]

on the whole system

\( M \vec{R} \)

contains terms like \( \vec{F}_{\alpha} \) which, of course, = 0
and terms like \( \sum_{\beta} \vec{F}_{\alpha \beta} \) and \( \sum_{\alpha} \vec{F}_{\alpha \beta} \) which, by Newton’s 3rd Law are equal and opposite.

\[ \therefore \text{This term} = 0 \]

\[ \therefore \quad \vec{F}_{\text{ext}} = M \vec{R} = \frac{d}{dt} \vec{P} \]
From the above equations, we can write 3 important results: -

I. Regardless of (possibly complicated) internal motions, the CM of a system obeys Newton's 2nd Law with only external forces.

II. The linear momentum of a system is the same as if a single particle of mass \( M \) were located at the position of the CM and moving as the CM moves.

III. The total linear momentum for a system that has no external forces is a constant and is equal to the linear momentum of the CM.
For example, if a shell explodes in mid air (neglecting air resistance)

The CM of the fragments continues to follow a parabolic trajectory (until one fragment hits the ground).

**Angular Momentum of a System**

There are lots of direct comparisons between linear motion and angular motion. For example:

- velocity $\leftrightarrow$ ang. velocity
- momentum $\leftrightarrow$ ang. mom
- force $\leftrightarrow$ torque etc.

so we can draw similar results for rotating systems of particles that we just derived for translational movement. I'll summarize the definitions and conclusions (see section 9.4).
It is often more convenient to describe a position vector with respect to the CM: 
\[ \mathbf{r}_a = \mathbf{R} + \mathbf{r}_a' \]

Then the angular momentum of the $\alpha$ particle is 
\[ \mathbf{L}_\alpha = \mathbf{r}_\alpha \times \mathbf{p}_\alpha \]

Summing over $\alpha$ gives: 
\[ \mathbf{L} = \sum\mathbf{L}_\alpha = \sum (\mathbf{r}_\alpha \times \mathbf{p}_\alpha) = \sum (\mathbf{r}_\alpha \times m_\alpha \mathbf{v}_\alpha) \]

Substituting for $\mathbf{r}_\alpha = \mathbf{R} + \mathbf{r}_\alpha'$ gives: 
\[ \mathbf{L} = \sum (\mathbf{r}_\alpha' + \mathbf{R}) \times m_\alpha (\mathbf{r}_\alpha' + \mathbf{R}) \]

Because $\sum m_\alpha \mathbf{r}_\alpha' = 0$, this can be reduced to 
\[ \mathbf{L} = \sum m_\alpha \mathbf{R} \times \mathbf{R} + \sum m_\alpha \mathbf{r}_\alpha' \times \mathbf{r}_\alpha' \]
\[ \mathbf{\vec{L}} = M \mathbf{\vec{R}} \times \mathbf{\vec{R}} + \sum_{x} m_{x} \mathbf{\vec{R}} \times \mathbf{\vec{R}}_{x} \]

\[ \mathbf{\vec{L}} = \mathbf{\vec{R}} \times \mathbf{\vec{p}} + \sum_{x} m_{x} \mathbf{\vec{R}} \times \mathbf{\vec{p}}_{x} \]

**IV** The total angular momentum about an origin is the sum of the angular momentum of the CM about that origin and the angular mom. of the system about the CM.

What is the behavior of angular momentum under the influence of external forces (torque)? We find
\[ \mathbf{\vec{L}} = \sum_{x} (\mathbf{\vec{R}} \times \mathbf{\vec{F}}_{x}^{\text{ext}}) = \sum_{x} \mathbf{\vec{r}}_{x} \times \mathbf{\vec{F}}_{x} = \mathbf{\vec{\tau}} \]

This is the rotational equivalent of Newton's 2nd Law.

**V** If the net resultant external torque about a given axis is zero, then the total angular momentum of the system about that axis remains constant.

**VI** Note: the total internal torque must vanish if \( \mathbf{\vec{F}}_{ab} = -\mathbf{\vec{F}}_{ba} \).
Energy of a System

Consider the work done on a system of particles in moving it from configuration 1 to configuration 2:

\[ W_{12} = \sum_{\alpha} \int \mathbf{F}_{\alpha} \cdot d\mathbf{r}_{\alpha} = T_{2} - T_{1} \]

where \[ T = \sum_{\alpha} \frac{1}{2} m_{\alpha} \mathbf{v}_{\alpha}^{2} \]

Again we transform into the CM frame

\[ \mathbf{r}_{\alpha} = \mathbf{R} + \mathbf{r}_{\alpha}' \]

\[ \mathbf{v}_{\alpha} = \mathbf{v} + \mathbf{v}_{\alpha}' \]

\[ T = \sum_{\alpha} \frac{1}{2} m_{\alpha} (\mathbf{v} + \mathbf{v}_{\alpha}')^{2} = \frac{1}{2} M \mathbf{v}^{2} + \sum_{\alpha} \frac{1}{2} m_{\alpha} \mathbf{v}_{\alpha}'^{2} + \mathbf{v} \cdot \sum_{\alpha} m_{\alpha} \mathbf{v}_{\alpha}' \]

The total kinetic energy of the system is equal to the sum of the kinetic energy of the total mass moving with the velocity of the CM + the kinetic energy of the individual particles moving relative to the CM.
Similarly the potential energy of the system can be written in terms of an external potential \( (\text{due to the external force}) \) and an internal potential \( (\text{due to the internal forces between the particles}) \).

\[
\vec{F}_{\text{ext}} = -\nabla_\alpha U_\alpha \quad \vec{F}_{\text{int}} = -\nabla_\alpha \vec{U}_{\alpha \beta}
\]

\[
U = \sum U_\alpha + \sum \vec{U}_{\alpha \beta}
\]

Then \( T_1 + U_1 = T_2 + U_2 \Rightarrow \text{conservation of energy of the system} \)

(If the system is rigid and the individual particles maintain their relative positions, then the internal potential energy remains constant and can be ignored when computing the total potential energy of the system.)