COLLISIONS

During a collision between 2 objects (hammer on nail, bouncing ball on floor, 2 billiard balls, etc.) the forces involved act over a very short time.

Of course, Newton's 2nd Law still holds

\[ \vec{F} = \frac{d\vec{P}}{dt} \]

\[ \therefore \vec{P}(t) - \vec{P}(0) = \int_{0}^{t} \vec{F}(t) \, dt \]

the Impulse

Often the forces are time-dependent, unmeasurable, and only act over an infinitesimal time. The impulse is defined and measured experimentally by the change in momentum.
In a collision:

\[ \begin{array}{c}
\text{1} \quad \rightarrow \quad \leftarrow \text{3}
\end{array} \]

At the instant of collision

\[ \vec{F}_{21} = -\vec{F}_{12} \]

\[ \frac{\Delta \vec{p}_1}{\Delta t} = -\frac{\Delta \vec{p}_2}{\Delta t} \]

\[ \therefore \Delta (\vec{p}_1 + \vec{p}_2) = 0 \]

\[ \Rightarrow \vec{p}_1 + \vec{p}_2 = \text{constant} \]

**Conservation of momentum**
Elastic Collisions
(Take particle #2 to be initially at rest.)

Initial

Final

\[ \vec{P}_i = \vec{P}_f \]
\[ m_1 \vec{u}_i = m_1 \vec{v}_i + m_2 \vec{v}_2 \]

x-component: \( m_1 u_i = m_1 v_i \cos \psi + m_2 v_2 \cos \xi \) \hspace{1cm} (1)

y-component: \( 0 = m_1 v_i \sin \psi - m_2 v_2 \sin \xi \) \hspace{1cm} (2)

For elastic collisions only we can also use conservation of kinetic energy:
\[ \frac{1}{2} m_1 u_i^2 = \frac{1}{2} m_1 v_i^2 + \frac{1}{2} m_2 v_2^2 \] \hspace{1cm} (3)

The "variables" are \( m_1, m_2, u_i, v_i, v_2, \psi, \xi \). 3 equations so we need to know 4 of the 7 variables to find the others.
Special case of equal masses:

\[ m_1 = m_2 = m \]

Then conservation of momentum \( \Rightarrow \)
\[ m \vec{u}_1 = m \vec{v}_1 + m \vec{v}_2 \Rightarrow \vec{u}_1 = \vec{v}_1 + \vec{v}_2 \]

and conservation of KE \( \Rightarrow \)
\[ \frac{1}{2} m u_1^2 = \frac{1}{2} m v_1^2 + \frac{1}{2} m v_2^2 \Rightarrow u_1^2 = v_1^2 + v_2^2 \]

which, in this case, is the Pythagorean theorem

\[ \psi + \phi = 90^\circ \]
Elastic Collisions in the Lab and CM Frames

**Lab Frame**
\[ \begin{align*} 
\text{Initial} & \quad m_1 \vec{u}_1 \\
& \quad \text{O} \rightarrow \triangleleft \quad m_2 \\
& \quad \text{O} \rightarrow \text{u}_2 = 0 \\
\end{align*} \]

**CM Frame**
\[ \begin{align*} 
\text{Initial} & \quad m_1 \vec{u}_1' \quad u_2' \quad m_2 \\
& \quad \text{O} \rightarrow \triangleleft \\
\end{align*} \]

**Final**
\[ \begin{align*} 
& \quad \vec{V}_2 \\
& \quad \vec{V}_1 \quad \theta \\
& \quad \vec{V}_2' \quad \vec{V}_1' \\
\end{align*} \]

The velocity \( \vec{v} \) of the CM is found from the definition of the CM:

\[ m_1 \vec{r}_1 + m_2 \vec{r}_2 = (m_1 + m_2) \vec{R} \]
\[ \therefore m_1 \vec{u}_1 + m_2 \vec{u}_2 = (m_1 + m_2) \vec{V} \]

Since \( \vec{u}_2 = 0 \) \( \Rightarrow \)

\[ \vec{V} = \frac{m_1 \vec{u}_1}{m_1 + m_2} \]

The final velocity \( (\vec{V}_1) \) in the lab is the final velocity in the CM \( (\vec{V}_1') \) + CM velocity \( (\vec{V}) \)
The nature of this vector diagram depends on the relative size of $\vec{V}_i'$ and $\vec{V}$.

If $\vec{V}_i'$ is greater than $\vec{V}$, there is only one possible solution for $\vec{V}_i'$ and $\theta$.

If $\vec{V}$ is greater than $\vec{V}_i'$, there are 2 possible solutions for $\vec{V}_i'$ and $\theta$.

The **Forward** solution and the **Backward** solution.
Quantitative Analysis

\[ \vec{V} = \frac{m_1 \vec{u}_1}{m_1 + m_2} \]

Since the velocity of particle 2 in the lab is initially zero ($\vec{u}_2 = 0$) \(\Rightarrow\)

In the CM: \[ \vec{u}'_2 = -\vec{V} \]

\[ \text{.: momentum of particle 2 in the CM is } -m_2 \vec{V} = -m_2 \frac{m_1 \vec{u}_1}{m_1 + m_2} \]

Velocity of particle 1 in the lab is \(\vec{u}_1\), so in the CM \[ \vec{u}'_1 = \vec{u}_1 - \vec{V} \]

\[ \text{.: momentum of particle 1 in the CM is } m_1 \vec{u}'_1 = \frac{m_2 \vec{u}_1}{m_1 + m_2} \]

\(\Rightarrow\) The total momentum in the CM frame is 0.
Initially the particles are directed towards each other. After the collision, they move in opposite directions in the CM.

For an elastic collision (conservation of momentum and kinetic energy) and, of course, the masses don’t change so the speeds before and after the collision are equal

\[ v_1' = u_1' = \frac{m_2 u_1}{m_1 + m_2} \]

\[ v_2' = u_2' = \frac{m_1 u_1}{m_1 + m_2} \]
The vector diagram relating $\mathbf{v}_i$, $\mathbf{v}_i'$ and $\mathbf{v}$

\[ \Rightarrow \quad v_i' \sin \theta = v_i \sin \psi \quad (1) \]

and \[ v_i' \cos \theta + v = v_i \cos \psi \quad (2) \]

Dividing (1) by (2) \[ \Rightarrow \]

\[ \tan \psi = \frac{\sin \theta}{\cos \theta + \frac{v}{v_i'}} \]

and, remembering that \[ v = \frac{m_i u_i}{m_1 + m_2} \]

and \[ v_i' = \frac{m_2 u_i}{m_1 + m_2} \]

\[ \Rightarrow \]

\[ \tan \psi = \frac{\sin \theta}{\cos \theta + \frac{m_i}{m_2}} \]
Limiting cases:

If \( m_1 \gg m_2 \) \( \Rightarrow \)
\[
\tan \psi = 0 \Rightarrow \psi = 0
\]
so mass \( m_1 \) is barely deflected.

If \( m_1 \ll m_2 \) \( \Rightarrow \)
\[
\tan \psi = \tan \Theta \Rightarrow \psi = \Theta
\]
so the lab and the CM frames are nearly the same (particle \( m_2 \) is unaffected by the collision).
If \( m_1 = m_2 \) \( \Rightarrow \tan \psi = \frac{\sin \Theta}{\cos \Theta + 1} \)

Writing \( \sin \Theta = 2 \sin \frac{\Theta}{2} \cos \frac{\Theta}{2} \)

and \( \cos \Theta = 2 \cos^2 \frac{\Theta}{2} - 1 \)

\( \Rightarrow \tan \psi = \frac{2 \sin \frac{\Theta}{2} \cos \frac{\Theta}{2}}{2 \cos^2 \frac{\Theta}{2}} = \tan \frac{\Theta}{2} \)

\( \therefore \psi = \frac{\Theta}{2} \)

The lab scattering angle is \( \frac{1}{2} \) the CM scattering angle. Since the maximum value for the CM scattering angle is \( \Theta = 180^\circ \)

\( \Rightarrow \) the maximum value for lab scattering (for \( m_1 = m_2 \)) is \( \psi = 90^\circ \).
We can also look at the vector diagram relating the lab and CM angles for particle 2.

\[ V_2 \sin \phi = V_2' \sin \theta \]
\[ V_2 \cos \phi = V - V_2' \cos \theta \]

Dividing (3) by (4) \[ \tan \phi = \frac{\sin \theta}{V/V_2' - \cos \theta} \]

Since \[ V = \frac{m_1 u_1}{m_1 + m_2} \] and \[ V_2' = \frac{m_2 u_1}{m_1 + m_2} \] \[ \Rightarrow V = V_2' \]

\[ \therefore \tan \phi = \frac{\sin \theta}{1 - \cos \theta} = \cot \frac{\theta}{2} = \tan \left( 90^\circ - \frac{\theta}{2} \right) \]

\[ \therefore \phi = 90^\circ - \frac{\theta}{2} = 90^\circ - \left( \frac{180^\circ - \theta}{2} \right) = \frac{\theta}{2} \]

[Lab angle for particle 2 is \( \phi \), CM angle is \( \theta = 180^\circ - \theta \)]

If \( m_1 = m_2 \) \[ \phi = \frac{\theta}{2} \] so we can write \[ \phi = 90 - \psi \Rightarrow \psi + \phi = 90^\circ \] so lab angle between the 2 particles = \( 90^\circ \).
Inelastic Collisions

When 2 particles interact, kinetic energy can be conserved (elastic collisions), can be lost (in the form of heat, sound or (for nuclear or high energy collisions) new particles) or gained (masses being converted to energy).

We define \( Q = KE_{\text{after}} - KE_{\text{before}} \)

called the \( Q \) value

If \( Q = 0 \) \( \Rightarrow \) elastic collisions

\( Q > 0 \) \( \Rightarrow \) Exoergic , KE is gained

\( Q < 0 \) \( \Rightarrow \) Endoergic , KE is lost

Inelastic collisions are endoergic, some energy is lost. How much depends on the nature of the collision. This can be expressed in another way.
The ratio of the relative final velocity to the initial velocity is known as the coefficient of restitution

\[ E = \frac{|V_2 - V_1|}{|u_2 - u_1|} \]

For perfectly elastic collisions, \( E = 1 \)

For perfectly inelastic collisions, \( E = 0 \)

i.e. the 2 bodies finally have no relative velocity. They stick together. This is the maximum loss of KE.

\[ m_1 \quad u_1 \quad m_2 \quad m_1 + m_2 \quad \quad \quad \quad 0 \rightarrow V_1 = V_2 \]

Conservation of momentum \( \Rightarrow \)

\[ m_1 u_1 = (m_1 + m_2) V_1 \]

\[ \therefore V_1 = \frac{m_1 u_1}{m_1 + m_2} \]

\[ [V_1 = V_2 = V, \text{ the CM velocity}] \]
Initial \( KE = KE_{\text{init}} = \frac{1}{2} m_1 u_i^2 \)

Final \( KE = \frac{1}{2} (m_1 + m_2) V_i^2 = \frac{1}{2} (m_1 + m_2) \frac{m_i^2 u_i^2}{(m_1 + m_2)^2} \)

\[ KE_{\text{final}} = \frac{1}{2} \frac{m_i^2 u_i^2}{(m_1 + m_2)} \]

\[ KE_{\text{lost}} = KE_{\text{init}} - KE_{\text{final}} = \frac{1}{2} m_i u_i^2 - \frac{1}{2} \frac{m_i^2 u_i^2}{(m_1 + m_2)} \]

\[ KE_{\text{lost}} = \frac{1}{2} m_i u_i^2 \left( \frac{m_i + m_2}{(m_1 + m_2)} - \frac{m_i}{(m_1 + m_2)} \right) = \frac{1}{2} \frac{m_1 m_2 u_i^2}{(m_1 + m_2)} \]

In the CM system:

\[ K_i = \frac{1}{2} m_1 u_i'^2 + \frac{1}{2} m_2 u_2'^2 \]

with \( u_i' = \frac{m_2 u_i}{m_1 + m_2} \) and \( u_2' = \frac{m_1 u_i}{m_1 + m_2} \)

\[ K_i = \frac{1}{2} \frac{m_1 m_2^2 u_i'^2}{(m_1 + m_2)^2} + \frac{1}{2} \frac{m_2 m_1^2 u_i'^2}{(m_1 + m_2)^2} \]
\[ K_i = \frac{1}{2} \frac{m_1 m_2 (m_1 + m_2) u_i^2}{(m_1 + m_2)^2} = \frac{1}{2} \frac{m_1 m_2 u_i^2}{m_1 + m_2} \]

\[ K_f = 0 \quad \text{since no relative motion} \]

\[ \therefore KE_{\text{lost}} = \frac{1}{2} \frac{m_1 m_2 u_i^2}{m_1 + m_2} \]

Recall that

\[ KE = \frac{1}{2} MV^2 + \sum_{\alpha} \frac{1}{2} m_{\alpha} v_{\alpha}^2 \]

\[ \uparrow \quad \downarrow \]

CM motion \quad motion with respect to the CM

In a fully inelastic collision, you can reduce the 2nd term to zero, but you can't change the 1st term.
Cross Sections

We use the scattering of beams of particles to study the nature of matter on the microscopic scale — atomic, nuclear or sub-nuclear.

If we have a beam intensity, $I$, which is the number of incident particles per second per area, then the number of particles scattered per second into a solid angle $d\Omega$ is

$$dN = \sigma(\theta) I d\Omega$$

where $\sigma(\theta)$ is the differential cross section for scattering through an angle $\theta$. 
\( \sigma(\theta) \) represents a probability of scattering through an angle \( \theta \). [Because I has dimensions of \( \sec^{-1} m^{-2} \), \( \sigma(\theta) \) has units of \( m^2 \) or area. That's why it's called a cross section.]

Particle \( m_1 \) approaches \( m_2 \) in such a way that, if there were no force between them, \( m_1 \) would pass \( m_2 \) with a distance of closest approach of \( b \).

\( b \) is called the impact parameter.
If having an impact parameter $b$ results in a scattering angle of $\Theta$ then the number scattered through $\Theta$ is

$$dN = I \cdot 2\pi b \cdot db$$

But from the definition of the cross section this is also $\sigma(\Theta)\Omega \, d\Omega = I \cdot 2\pi b \cdot db$

$$d\Omega = \frac{\text{Area}}{r^2} = (rd\theta) \cdot \frac{r \sin \Theta \, d\phi}{r^2}$$

$$\therefore d\Omega = \sin \Theta \, d\theta \, d\phi = 2\pi \sin \Theta \, d\Theta \text{ for azimuthal symmetry}$$

$$\therefore \sigma(\Theta) \cdot 2\pi \sin \Theta \, d\theta = 2\pi b \cdot db$$

$$\therefore \sigma(\Theta) = \frac{b}{\sin \Theta} \frac{db}{d\theta}$$
How is $b$ related to $\Theta$?

$m_1$ moves from an initial asymptotic angle of $\alpha$ to a final asymptotic angle of $\beta$. (The instantaneous angle is $\phi$.)

$\alpha = \beta$ (by symmetry) and $\alpha + \beta + \Theta = 180^o$

$\therefore 2\alpha + \Theta = 180^o \quad \text{or} \quad \alpha = 90 - \frac{\Theta}{2}$

Conservation of angular momentum $\Rightarrow$

$m_1 v_0 b = m_1 v r = m_1 (r \frac{d\phi}{dt}) r$

$\therefore \frac{d\phi}{dt} = \frac{v_0 b}{r^2}$

[$\phi$ and $r$ vary as a function of $t$]
Rutherford Scattering

We will take the specific example of the scattering of α particles (helium nuclei) by gold nuclei.

The repulsive force between the two particles is given by Coulomb's Law

\[ F = \frac{k \cdot q_1 q_2}{r^2} \]

where \( k = 8.99 \times 10^9 \text{ N.m}^2/\text{C}^2 \)

\( q_1 = Z_1 e \) (\( Z_1 = 2 \))

\( q_2 = Z_2 e \) (\( Z_2 = 79 \))

\( e = 1.6 \times 10^{-19} \text{ C} \)

The component of force perpendicular to the trajectory results in a change of momentum for the incident particle

\[ \Delta p = 2m_1 v_0 \sin \frac{\theta}{2} \]

where the velocity of mass \( m_1 \) is \( v_0 \)
\[ \int_{t=0}^{t=\infty} F \cos \phi \, dt = \Delta p \]

But \( \frac{d\phi}{dt} = \frac{V_0 b}{r^2} \) \hspace{1cm} \text{\therefore substituting for } dt \Rightarrow \]

\[ \int_{\phi_i}^{\phi_f} \frac{k g_1 g_2 \cos \phi}{V_0 b} \, d\phi = 2 m_1 V_0 \sin \frac{\theta}{2} \]

Also remember \( \alpha = 90 - \frac{\theta}{2} \) \hspace{1cm} \therefore \sin \alpha = \cos \frac{\theta}{2} \]

\[ \therefore \frac{k g_1 g_2 \cos \frac{\theta}{2}}{m_1 V_0^2 b} = \sin \frac{\theta}{2} \]
\[
\therefore b = \frac{k g_i g_2}{m_1 v_0^2} \cot \frac{\theta}{2} = \frac{1}{2} k g_i g_2 \cot \frac{\theta}{2} T_0
\]

where \( T_0 = \frac{1}{2} m_1 v_0^2 \)

We've got a bit more work to do so I'll write \( K = \frac{1}{2} k g_i g_2 \Rightarrow b = K \cot \frac{\theta}{2} \)

Remember (3 or 4 pages back!)

\[
\sigma(\theta) = \frac{b}{\sin \theta} \frac{db}{d\theta}
\]

\[
\therefore \frac{db}{d\theta} = \frac{K}{2} \frac{1}{\sin^2 \frac{\theta}{2}}
\]

\[
\therefore \sigma(\theta) = \frac{K \cot \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \frac{K}{2 \sin^2 \frac{\theta}{2}}
\]

\[
\sigma(\theta) = \frac{k^2 (z_1 - e)^3 (z_2 - e)^3}{16 \frac{1}{T^2}} \frac{1}{\sin^4 \frac{\theta}{2}}
\]
This famous formula, derived by Ernest Rutherford in 1912, was verified by the experiments of Geiger and Marsden. The important features are:

1. The scattering is proportional to the square of the atomic numbers of both the incident and target nuclei.
2. The scattering is inversely proportional to the square of the incident KE.
3. The scattering is inversely proportional to the $4^{th}$ power of half of the scattering angle.