Gravitation

Newton's Law of gravity:

\[ F_{21} = -F_{12} \]

\[ |F| = \frac{G m_1 m_2}{r^2} \]

where \( G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2 \)

Vector notation: Use the heavier mass \( M \) as the origin of the coordinate system.

Force on \( m \) is \( \vec{F} = -\frac{GMm^2}{r^2} \)

Note that the negative sign \( \Rightarrow \) opposite direction to the \( \vec{r} \) vector \( \Rightarrow \) attractive force.
Extended mass distributions:

\[
\vec{F} = -G m \int_{\text{vol}} \frac{\rho(\vec{r}) \, d\vec{v}'}{r^2}
\]

\[
M = \int_{\text{vol}} \rho(\vec{r}) \, d\vec{v}'
\]

**Gravitational Field**

We find it useful to define a force per unit mass, i.e. independent of \( m \)

\[
\vec{g} = \frac{\vec{F}}{m} = -G M \frac{\vec{r}}{r^2}
\]

For a single source mass \( M \)

or

\[
\vec{g} = -G \int_{\text{vol}} \frac{\rho(\vec{r}') \, d\vec{v}'}{r^2}
\]

for an extended source
Gravitational Potential Energy

The force of gravity is a conservative force \( \Rightarrow \int_{r_i}^{r_2} \mathbf{F} \cdot d\mathbf{r} \) is path independent.

\( \Rightarrow \mathbf{F} \) can be written in terms of the gradient of a scalar function \( \mathbf{F} = -\nabla U \)

![Diagram of gravitational potential energy]

We can define gravitational potential energy as

\[
U(r_2) - U(r_i) = -\int_{r_i}^{r_2} \mathbf{F} \cdot d\mathbf{r}
\]
\[ U(\vec{r}_2) - U(\vec{r}_1) = \int_{\vec{r}_1}^{\vec{r}_2} \frac{GMm}{r^2} \, d\vec{r} \]

\[ = GMm \int_{\vec{r}_1}^{\vec{r}_2} \frac{d\vec{r}}{r^2} \]

\[ = \frac{-GMm}{r_2} - \left( \frac{-GMm}{r_1} \right) \]

\[ \Rightarrow \quad U(\vec{r}) = -\frac{GMm}{r} \]

choosing the arbitrary constant of integration as \( = 0 \) so \( U(\vec{r}) \to 0 \) as \( r \to \infty \).
Spherical Coordinates

Spherical coordinates:
\[ dv = r^2 \sin \theta \, dr \, d\theta \, d\phi \]

\[ da = r^2 \sin \theta \, d\theta \, d\phi \]

\[ dv = r^2 \sin \theta \, dr \, d\theta \, d\phi \]

\[ d\vec{r} = dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi} \]

\[ \vec{V} = \frac{dV}{dt} \hat{r} + \frac{1}{r} \frac{d}{d\theta} V \hat{\theta} + \frac{1}{r \sin \theta} \frac{d}{d\phi} V \hat{\phi} \]
Gravitational Potential

Here we can also define a variable independent of $m$. This is the gravitational potential defined as the potential energy per unit mass

$$\phi(\vec{r}) = \frac{U(\vec{r})}{M} = \frac{-GM}{r}$$

for a single source mass $M$

or

$$\phi(\vec{r}) = -G \int_{\text{Vol}} \frac{\rho(\vec{r}') dV'}{r}$$

for an extended source

Also, because $\vec{F} = -\vec{\nabla}U$

we can write $\vec{g} = -\vec{\nabla}\phi$
The Shell Theorem

Find the gravitational potential and field for a thin spherical shell of radius \( r' \). All of the thin disk (outlined in red) is equidistant from point \( P \).

Mass of this disk is
\[
dM = \rho \cdot 2\pi r' \sin \theta \cdot r' \, d\theta \cdot dr'
\]
\[
= 2\pi \rho \pi r'^3 \, dr' \sin \theta \, d\theta
\]

Total mass of the thin shell is
\[
M = \rho \cdot 4\pi r'^2 \, dr'
\]

\[
\Rightarrow \frac{dM}{M} = \frac{2\pi \rho \pi r'^2 \, dr' \sin \theta \, d\theta}{4\pi \rho \pi r'^3 \, dr'} = \frac{\sin \theta \, d\theta}{2}
\]
\[ \phi = -GM \int \frac{dM}{r} = -GM \int_0^\frac{\pi}{2} \frac{\sin \theta}{r} d\theta = GM \int_0^\frac{\pi}{2} d\left( \frac{\cos \theta}{r} \right) \]

\[ r = \sqrt{r'^2 + R^2 - 2r'R \cos \theta} \]

\[ \therefore \phi = GM \int_0^\frac{\pi}{2} d(\cos \theta) \left( r'^2 + R^2 - 2r'R \cos \theta \right)^{-\frac{1}{2}} \]

\[ = GM \left[ \frac{1}{\left( \frac{1}{2} \right)} \left( r'^2 + R^2 - 2r'R \cos \theta \right)^{\frac{1}{2}} \frac{1}{(-2r'R)} \right]_0^\frac{\pi}{2} \]

\[ = -GM \left[ \left( r'^2 + R^2 + 2r'R \right)^{\frac{1}{2}} - \left( r'^2 + R^2 - 2r'R \right)^{\frac{1}{2}} \right] \]

\[ \therefore \phi = -GM \left[ \left( r' + R \right) - 1 + r' - R \right] \]

depends on which is larger i.e. if \( P \) is inside or outside the shell
If \( R > r' \) (i.e. \( P \) outside of shell)

\[
\phi = \frac{-GM}{2r'R} \left[ r' + R - R + r' \right] = \frac{-GM \cdot 2r'}{2r'R}
\]

\[\phi_{R>r'} = -\frac{GM}{R}\]

Same as if all of \( M \) were located at the center of shell.

If \( R < r' \) (i.e. \( P \) inside of shell)

\[
\phi = \frac{-GM}{2r'R} \left[ r' + R - r' + R \right] = \frac{-GM \cdot 2R}{2r'R}
\]

\[\phi_{R<r'} = -\frac{GM}{r'}\]

Independent of position of point \( P \) inside the shell.
\[ \Phi(R) = \begin{cases} 
\frac{-GM}{r'} & \text{for } R < r' \\
\frac{-GM}{R} & \text{for } R > r' 
\end{cases} \]

Then, using \( \vec{g} = -\vec{\nabla}\Phi \)

\[ \Rightarrow \quad \vec{g}(R) = -\frac{1}{2} \frac{\partial^2 \Phi}{\partial R^2} \hat{R} \]

\[ \therefore \quad \vec{g}(R) = \begin{cases} 
0 & \text{for } R < r' \\
\frac{-GM}{R^3} & \text{for } R > r' 
\end{cases} \]
**Lines of Force**

Similarly to electrostatics, we can think of the continuously connected vectors of $\mathbf{g}(\mathbf{r})$ pointing into the mass that produces them (like a negative electric charge).

![Diagram of lines of force]

The fact that the gravitational field gets larger as we get closer to the mass can be represented by the density of the lines (i.e., the number of lines passing through a unit area).
Poisson's Equation

Consider an arbitrary surface $S$ enclosing a mass $m$

Gravitational Flux $= \int_S \mathbf{g} \cdot d\mathbf{a} = \int_S \mathbf{g} \cdot \hat{n} \, da$

$= -\int_S \frac{Gm}{r^2} \hat{r} \cdot \hat{n} \, da = -6m \int_S \frac{\cos \Theta \, da}{r^2}$

But $\cos \Theta \, da = d\Omega$, a solid angle

So $\int_S \frac{\cos \Theta \, da}{r^2} = \text{total solid angle} = 4\pi$
So \( \text{Grav. Flux} = -4\pi Gm \)

If there are several masses enclosed by the surface \( \Rightarrow \)

\[ \text{Grav. Flux} = -4\pi G \sum m_i \]

If there is a continuous mass distribution enclosed by the surface then:

\[
\int_{S} \vec{\nabla} \cdot \vec{\phi} \, d\vec{a} = -4\pi G \int_{V} \rho dV 
\]

\( \text{Gauss' Law} \)

We can also make use of Gauss' divergence theorem:

\[
\int_{S} \vec{A} \cdot d\vec{a} = \int_{V} (\vec{\nabla} \cdot \vec{A}) \, dV
\]

\( \Rightarrow \)

\[
\int_{V} (\vec{\nabla} \cdot \vec{\phi}) \, dV = -4\pi G \int_{V} \rho dV
\]

\( \Rightarrow \)

\[
\vec{\nabla} \cdot \vec{\phi} = -4\pi G \rho
\]
Remember Gauss' Law for electrostatics:
\[ \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} = 4\pi k \rho \]

where \(\vec{E}\) = Electric field, \(\rho\) = charge density, \(\varepsilon_0\) = permittivity, \(k\) = Coulomb's constant.

Remember also that we can write:
\[ \vec{E} = -\nabla \phi \]

\[ -\nabla \cdot \nabla \phi = -4\pi G \rho \]

\[ \Rightarrow \nabla^2 \phi = 4\pi G \rho \]

Poisson's Equation

If there are no masses inside the surface we have:
\[ \nabla^2 \phi = 0 \]

Laplace's Equation
An Example

Calculate the force on a mass \( m \) which is situated a distance \( z \) above a disk of radius \( a \) and mass \( M \).

\[
\text{density} \ \rho \ \text{kg/m}^2 \\
\Rightarrow M = \pi a^2 \rho
\]

Force due to the thin ring

\[
dF = \frac{GmdM}{r^2}
\]

Of course the contribution from the other side of the thin ring will cancel out the horizontal direction but will add in the \( z \) direction.
We calculate the $z$ component

$$dF_z = -Gm \frac{dM \cos \theta}{r^3}$$

where $dM = 2\pi x \, dx \, \rho$.

and $\cos \theta = \frac{z}{r}$

$$\Rightarrow dF_z = -Gm \rho \frac{2\pi x \, dx \cdot z}{r^3}$$

$$\therefore F_z = -Gm \rho \cdot 2\pi z \int_0^a \frac{x \, dx}{(z^2 + x^2)^{3/2}}$$

$$\therefore F_z = -Gm \rho \cdot 2\pi z \left[\frac{1}{(z^2 + x^2)^{1/2}}\right]^a_0$$

$$\therefore F_z = -Gm \rho \cdot 2\pi z \left[\frac{1}{z} - \frac{1}{(z^2 + a^2)^{1/2}}\right]$$

$$\Rightarrow F_z = -Gm \rho \cdot 2\pi \left[1 - \frac{z}{\sqrt{z^2 + a^2}}\right]$$
Alternative method using potential

Potential due to thin ring, \( d\phi = -\frac{G\,dM}{r} \)

(note: no problems due to vectors)

As before \( dM = 2\pi \rho x\,dx \)
\[ d\phi = -\frac{2\pi \rho G x\,dx}{r} = -\frac{2\pi \rho G x\,dx}{(x^2 + z^2)^{\frac{1}{2}}} \]

\[ \therefore \phi(z) = -2\pi \rho G \int_0^a \frac{x\,dx}{(x^2 + z^2)^{\frac{1}{2}}} \]

\[ = -2\pi \rho G \left[ (x^2 + z^2)^{\frac{1}{2}} \right]_0^a \]
\[ \phi(z) = -2\pi \rho G \left[ \frac{z}{\sqrt{z^2 + a^2}} - z \right] \]

Then \[ F = -\nabla U = -m \nabla \phi \]

\[ \Rightarrow F_x = -m \frac{\partial \phi}{\partial z} = +2\pi m \rho G \left[ \frac{z}{\sqrt{z^2 + a^2}} - 1 \right] \]

As before
Equipotential Surfaces

The potential function $\phi(\vec{r})$ is defined at every point in space. So the equation $\phi(\vec{r}) = \text{constant}$ defines a surface in which the potential is constant — called an equipotential surface [.....dubhh!]

The gravitation field vector is given by $\vec{g} = -\nabla \phi$ so $\vec{g}$ can have no component along the equipotential surface $\Rightarrow$ every line of force must be normal to every equipotential surface. Equipotential surfaces around a single isolated point mass are spheres.
Two point masses \( M \) separated by a distance \( d \) would give the equipotential surfaces shown.

\[ \phi = -GM \left( \frac{1}{r_1} + \frac{1}{r_2} \right) = \text{constant} \]

defines the equipotential surfaces.
Lides

The book devotes 6½ pages to a discussion of tidal forces (mostly due to the gravitational attraction of the moon).

We have to calculate the forces on a loose object on the surface of the earth (the oceans) relative to the force on the earth itself.
There are ~2 tides per day because water closest to moon is attracted away from the Earth and the Earth is attracted away from the water that is farthest from the moon.
Tidal forces are caused by the difference between the moon's gravitational pull at the center of the Earth and on the Earth's surface. This is due both to a slight difference in distance and/or a slight difference in direction.
The acceleration on mass \( m \) at the Earth's surface due to the moon is
\[
\ddot{r}_1 = -\frac{GM_m\hat{R}}{R^2}
\]
where \( M_m \) is the mass of the moon.

The acceleration of the Earth (as a whole) due to the moon is
\[
\ddot{r}_2 = -\frac{GM_m\hat{D}}{D^2}
\]

The difference gives rise to a tidal force
\[
\vec{F}_T = m\ddot{r} = m(\ddot{r}_1 - \ddot{r}_2)
\]
\[
= -GM_m m\left(\frac{\hat{R}}{R^2} - \frac{\hat{D}}{D^2}\right)
\]
We can evaluate this force at different points on the surface of the Earth relative to the moon.

For example, at point a, farthest from the moon, both vectors \( \hat{R} \) and \( \hat{D} \) are pointing in the same direction along the x-axis away from the moon. And, obviously, \( R > D \).

\[
F_{Tx} = -GMm \left( \frac{1}{(D+r)^2} - \frac{1}{D^2} \right)
\]
\[ F_{Tx} = -\frac{GM_{m}m}{D^2} \left( (1 + \frac{r}{D})^{-2} - 1 \right) \]

\[ \approx -\frac{GM_{m}m}{D^2} \left( 1 - \frac{2r}{D} + \frac{3r^2}{D^3} + \ldots - 1 \right) \]

\[ F_{Tx} \approx +\frac{GM_{m}m2r}{D^3} \]

(Note: in positive x direction, away from the moon)

At location b, closest to the moon, R < D and we can write

\[ F_{Tx} = -\frac{GM_{m}m}{D^2} \left( \frac{1}{(D+r)^2} - \frac{1}{D^2} \right) \]

\[ = -\frac{GM_{m}m}{D^2} \left( (1 - \frac{r}{D})^{-2} - 1 \right) \]

\[ \approx -\frac{GM_{m}m2r}{D^3} \] (in negative x direction, towards the moon)
At location \( C \), \( R = D \) but there is a small component of \( \hat{R} \) in the y direction.

\[
\theta = \frac{r}{D}
\]

\[
\therefore F_{Ty} = -\frac{GM_{m}m}{D^2} \frac{r}{D} = -\frac{GM_{m}m r}{D^3}
\]

(Points in the negative y direction, towards the center of the Earth.)

At an arbitrary point on the Earth's surface

We can write:

\[
F_{Tx} = 2\frac{GM_{m}m r \cos \theta}{D^3}
\]

and

\[
F_{Ty} = -\frac{GM_{m}m r \sin \theta}{D^3}
\]
How high are the tides caused by these forces? Newton made a "guestimate". He suggested a calculation that imagined digging two wells to the center of the Earth — one in the direction of high tide (our $x$ axis) and one in the direction of low tide (our $y$ axis).

Then we can calculate the work done to move a mass $m$ from $c$ to $a$ via the wells.
\[ W = \int_{r}^{0} F_{Ty} \, dy + \int_{0}^{r} F_{Tx} \, dx \]

\[ = \frac{GM_{mm}}{D^3} \left[ \int_{r}^{0} y \, dy + 2 \int_{0}^{r} x \, dx \right] \]

\[ = \frac{GM_{mm}}{D^3} \left[ \frac{1}{2} r^2 + r^2 \right] = \frac{3}{2} \frac{GM_{mm} r^2}{D^3} \]

We can set this equal to the tidal height difference, \( mgh \Rightarrow \)

\[ \frac{3}{2} \frac{GM_{mm} r^3}{D^3} = mgh \quad \Rightarrow \quad h = \frac{3}{2} \frac{GM_{mm} r^3}{g D^3} \]

Substituting:
\[ G = 6.67E-11 \quad M_m = 7.35E22 \text{ kg} \]
\[ r = 6.37E6 \text{ m} \quad D = 3.84E8 \text{ m} \]

\[ \Rightarrow \quad h = \frac{3}{2} \frac{(6.67E-11)(7.35E22)(6.37E6)}{9.81 \cdot (3.84E8)^3} \]

\[ \Rightarrow \quad h = 0.54 \text{ m} \]
Note: we can also estimate the relative tidal effects of the sun.
Forces (and Newton's guesstimate of height) both \( \Rightarrow \sim \frac{M}{D^3} \)

\[ \therefore \frac{\text{Effect of Sun}}{\text{Effect of moon}} = \left(\frac{\frac{M_s}{M_m}}{\frac{D_s}{D_m}}\right)^3 \]

\[ = \left(\frac{1.99 \times 10^{30}}{7.35 \times 10^{22}}\right) \div \left(\frac{1.50 \times 10^{11}}{3.84 \times 10^{22}}\right)^3 \]

\[ = \frac{2.71 \times 10^7}{(391)^3} = \frac{2.71 \times 10^7}{5.96 \times 10^7} \]

\[ = 0.46 \text{ which is a sizeable effect} \]

\( \Rightarrow \) tides are stronger when sun and moon line up (full moon and new moon) and weaker when they are at right angles to one another (first and third quarters).